QUASI SLATER AND FM QUALIFICATIONS FOR SEMI-INFINITE PROGRAMMING WITH APPLICATIONS

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Abstract. The well-known Farkas-Minkowski (FM) type qualification plays an important role in linear semi-infinite programming, and has been extensively developed by many authors in establishing optimality conditions, duality and stability for semi-infinite programming. In this paper, we introduce the concept of quasi Slater condition for semi-infinite convex inequality system and present that the Slater type conditions imply the FM qualification under some appropriate continuity assumption of the set-valued mapping $i \mapsto f_i(x)$. Applying these relationships, we establish dual characterizations, both asymptotic and nonasymptotic, for set containment problems and provide some sufficient conditions for ensuring the strong Lagrangian duality and Farkas lemma.

Key words. Slater condition, FM qualification, semi-infinite programming, set containment characterization, strong Lagrangian duality, Farkas lemma.

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1. Introduction. Many problems in optimization and approximation theory can be expressed into the following one/two types: one is a system of convex inequalities

$$x \in C; f_i(x) \le 0 \quad \text{for each } i \in I, \tag{1.1}$$

and the other is a minimization problem

$$\begin{array}{ll} \text{Minimize} & f(x), \\ \text{subject to} & f_i(x) \le 0,\ i \in I, \\ & x \in C, \end{array} \tag{1.2}$$

where $C \subseteq \mathbb{R}^n$ is a closed convex set, I is an arbitrary (possibly infinite) index set, $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is a proper convex function, and each $f_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function. These two problems have been studied extensively by many authors under various degrees of generality imposed on the index set I, the system of functions $\{f_i : i \in I\}$, or on the underlying space; see e.g. [2, 3, 7, 17, 18, 21, 25, 27, 28, 30, 33] and references therein.

Constraint qualifications involving epigraphs play an important role in optimization and have been widely studied and extensively developed by many authors; see e.g. [2, 3, 6, 7, 11, 12, 15, 24, 30, 31, 32]. One of the most important constraint qualifications is the Farkas-Minkowski (FM) type qualification, which was originally introduced in [4] for the linear system in the Euclidian space to study the duality theory. Later Dinh et al. in [7] extended the FM qualification for the convex system $\{C; f_i : i \in I\}$ in a locally

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convex Hausdorff topological vector space, and Jeyakumar et al. in [22] proposed the FM qualification for the cone system in a Banach space under the name of closed cone constraint qualification (CCCQ) to establish optimality conditions, duality theorems and stability theorems. Furthermore, the FM qualification was extended by Li et al. in [31] for the more general case when the involved functions are not necessarily lower semicontinuous (in short, lsc), where they named it the conical epigraph hull property (conical EHP), which is equivalent to the FM qualification considered in [7] when the involved functions are lsc. The relationships between the conical EHP and other well-known constraint qualifications, including the basic constraint qualification (BCQ) for system (1.1), the sum of epigraphs constraint qualification (SECQ) and the strong conical hull intersection property (CHIP) for the system $\{C, C_i : i \in I\}$ of closed convex sets were also studied in [30, 31]. These constraint qualifications play an important role not only in optimization but also in many other areas, such as best approximation theory and conic programming. For more details about the BCQ, SECQ and strong CHIP, one can refer to [9, 10, 18, 26, 27, 28, 29, 30, 31, 33].

In recent years, much attention has been given to establish and develop sufficient conditions for ensuring constraint qualifications. One of the most well-known conditions is the Slater type condition. The Slater condition was introduced in [33] for the semi-infinite convex inequality system in the Euclidean space. For more general case, Li and Ng in [28] introduced the concepts of Slater condition and weak Slater condition in a Banach space and provided some sufficient conditions to ensure the BCQ for system (1.1) under the weak Slater condition. For a cone convex system, Jeyakumar et al. in [22] extended the generalized Slater type conditions (i.e., the quasi-relative interior conditions) and showed that the generalized Slater type conditions imply the CCCQ. Considering the special case of system (1.1) when each f_i is the indicator function of a closed convex set C_i in a Banach space, Li et al. in [29] and [30] provided some interior-type sufficient conditions for the system $\{C, C_i : i \in I\}$ of closed convex sets to satisfy the strong CHIP and the SECQ, respectively. However, to our knowledge, few papers are devoted to discussing when the system (1.1) satisfies the FM qualification.

The main objective of this paper is to establish some sufficient conditions of the FM qualification in the Euclidian space. Recall that the FM qualification implies the locally FM qualification, see [7, 15, 31] for details, and it was shown in [7] that when the sup-function of $\{f_i : i \in I\}$ is continuous, the locally FM qualification is equivalent to the BCQ. Thus, it is natural to inquire whether the sufficient conditions, originally proposed in [28] to ensure the BCQ, can imply the FM qualification. Considering the system (1.1), we show in Section 3 that under an appropriate continuity assumption of the set-valued mapping $i \mapsto f_i(x)$, both the Slater condition and the weak Slater condition can imply the FM qualification. Note that the weak Slater condition requires the active constraints to be finite. To meet much broader class of problems, inspired by a closure condition in [17], we introduce the concept of quasi Slater condition to cover the case when the active constraints are infinite, which is much weaker than the weak Slater condition. Moreover, we demonstrate that the quasi Slater condition also implies the FM qualification under the same continuity assumption. These results extend and improve the corresponding results in [28] in the finite dimensional space.

The motivation of our work stems from various applications. One is the set containment characterization, which has many applications in several important problems, such as optimization, mathematical programming and knowledge-based data classification; see e.g. [13, 14, 16, 17, 23, 34, 35]. Here we specially mention the works of Mangasarian [34] and Jeyakumar [21]. In [34], Mangasarian provided nonasymptotic dual characterizations of the containment of a polyhedral set in another polyhedral set and the containment of a closed convex set in a reverse-convex set, defined by finitely many convex constraints. In [21], Jeyakumar established asymptotic dual characterizations of the containment of a closed convex set, defined by infinitely many convex constraints, in a reverse-convex/convex set, defined by finitely many convex constraints. Also, the author derived nonasymptotic dual characterizations of the set containment of a closed convex set, defined by infinitely many convex inequalities, in a polyhedral set under the Slater condition and an additional assumption that each linear functional generating the polyhedral set cannot be 0. In this paper, as applications, we establish nonasymptotic dual characterizations of the set containment in sets defined by infinitely many inequalities under some weaker assumptions, which extend and improve the corresponding results in [21].

The other is the Lagrangian duality and Farkas type results for problem (1.2), which are fundamental in convex optimization and other fields, such as game theory, set containment problems, etc. The literature on these areas is very rich (see e.g. [6, 7, 12, 17, 20, 21, 33]), but few for semi-infinite programming except the work of Fang et al. [12], where they characterized the Lagrangian duality and Farkas lemma for semi-infinite programming in terms of epigraphs of the conjugate functions of the involved functions. In this paper, applying our main results on FM qualification for system (1.1), we establish some sufficient conditions for ensuring the strong Lagrangian duality and Farkas lemma, which seem new as far as we know.

The paper is organized as follows. In Section 2, we present the notations and preliminary results used in this paper. In Section 3, we establish some sufficient conditions to ensure the FM qualification for system (1.1). Applications to the set containment characterization, strong Lagrangian duality and Farkas lemma are investigated in section 4.

2. Notations and preliminary results. The notation used in the present paper is standard (cf. [18, 19]). In particular, we consider the n-dimensional Euclidian space $\mathbb{R}^n$ and use $\langle x^*, x\rangle$ to denote the inner product of $\mathbb{R}^n$. As usual, we use $\mathbb{R}_+$ to denote the subset of $\mathbb{R}$ consisting of all nonnegative real numbers. For a subset $Z \subseteq \mathbb{R}^n$, the interior (resp., relative interior, closure, convex cone hull, linear hull, affine hull) of Z is denoted by $\mathrm{int}Z$ (resp., $\mathrm{ri}Z$, $\mathrm{cl}Z$, $\mathrm{cone}Z$, $\mathrm{span}Z$, $\mathrm{aff}Z$), and the orthogonal supplement of Z is denoted by $Z^{\perp}$ if Z is a subspace. We shall also adopt the convention that $\mathrm{cone}Z = \{0\}$ when Z is an empty set. The normal cone of Z at $z_0 \in Z$ is denoted by $N_Z(z_0)$ and defined by

$$N_Z(z_0) := \{x^* \in \mathbb{R}^n : \langle x^*, z - z_0\rangle \le 0 \quad \text{for all } z \in Z\}.$$

The indicator function δ_Z of Z is defined by

$$\delta_Z(x) := \begin{cases} 0, & x \in Z, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a proper convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the domain of f is denoted by $\mathrm{dom}f := \{x \in \mathbb{R}^n : \ f(x) < +\infty\}$. Then the subdifferential of f at $x \in \mathrm{dom}f$, denoted by $\partial f(x)$, is defined by

$$\partial f(x) := \{x^* \in \mathbb{R}^n : f(x) + \langle x^*, y - x\rangle \le f(y) \quad \text{for all } y \in \mathbb{R}^n\}.$$

The epigraph and conjugate of a function f on $\mathbb{R}^n$, denoted by $\mathrm{epi}f$ and f^*, are defined respectively by

$$\mathrm{epi}f := \{(x, r) \in \mathbb{R}^{n+1} : f(x) \le r\},$$

and

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : \ x \in \mathbb{R}^n\} \quad \text{for each } x^* \in \mathbb{R}^n.$$

If f and h are proper lsc convex functions on $\mathbb{R}^n$, then we have

$$f \le h \Longleftrightarrow f^* \ge h^* \Longleftrightarrow \operatorname{epi} f^* \subseteq \operatorname{epi} h^*. \tag{2.1}$$

In particular, for closed convex sets A and B, the following assertions are well-known:

$$N_A(x) = \partial \delta_A(x) \quad \text{for each } x \in A,$$

and

$$\operatorname{epi}\delta_A^* \subseteq \operatorname{epi}\delta_B^* \Longleftrightarrow A \supseteq B. \tag{2.2}$$

Let $\{A_i : i \in J\}$ be a system of subsets of $\mathbb{R}^n$. The set $\sum_{i\in J} A_i$ is defined by

$$\sum_{i\in J} A_i := \begin{cases} \left\{\sum_{i\in J_0} a_i : a_i \in A_i, J_0 \subseteq J \text{ being finite}\right\}, & J \ne \emptyset, \\ \{0\}, & J = \emptyset. \end{cases}$$

In particular, we adopt the convention that $\sum_{i\in J} a_i = 0$ if $J = \emptyset$.

The following lemma characterizes the epigraph of the conjugate of the sum of two functions; see [12, Lemma 2.1] and [1, Corollary 2.2].

LEMMA 2.1. *Let $f, h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. Then*

$$\operatorname{epi}(f+h)^* \supseteq \operatorname{epi} f^* + \operatorname{epi} h^*. \tag{2.3}$$

Furthermore, the equality holds:

$$\operatorname{epi}(f+h)^* = \operatorname{epi} f^* + \operatorname{epi}\ h^*, \tag{2.4}$$

if $\operatorname{int}(\operatorname{dom} f) \cap \operatorname{dom} h \ne \emptyset$ *or* $\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} h) \ne \emptyset$.

3. The Slater and FM qualifications. Throughout this paper, we consider a proper convex objective function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and the convex system $\{C; f_i : i \in I\}$, where I is an (possibly infinite) index set, $C \subseteq \mathbb{R}^n$ is a closed convex set, and each $f_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function. The sup-function of the system $\{f_i : i \in I\}$ is denoted by F and defined by

$$F(x) := \sup_{i\in I} f_i(x) \quad \text{for each } x \in \mathbb{R}^n. \tag{3.1}$$

We always assume for the whole paper that F is finite (and so continuous) on $\mathbb{R}^n$, and use A to denote the solution set of system (1.1):

$$A := C \cap S, \tag{3.2}$$

where S is denoted by

$$S := \{x \in \mathbb{R}^n : f_i(x) \le 0 \text{ for all } i \in I\} = \{x \in \mathbb{R}^n : F(x) \le 0\}. \tag{3.3}$$

Moreover, to avoid the triviality in our study for (1.2), we always assume that $\operatorname{dom} f \cap A \ne \emptyset$.

Given $x_0 \in \mathbb{R}^n$, we use $I(x_0)$ to denote the active index set at x_0, defined by

$$I(x_0) := \{i \in I : f_i(x_0) = 0\}. \tag{3.4}$$

For convenience, we set

$$I_0 = I \setminus I(x_0), \tag{3.5}$$

and use F_0 to denote the sup-function of the subsystem $\{f_i : i \in I_0\}$, that is,

$$F_0(x) := \sup_{i \in I_0} f_i(x) \quad \text{for each } x \in \mathbb{R}^n. \tag{3.6}$$

In the following definition, we introduce the concepts of Slater type conditions for the system $\{f_i : i \in I\}$ considered here.

Definition 3.1. *Let D and C be two convex subsets of $\mathbb{R}^n$. The system $\{f_i : i \in I\}$ is said to satisfy*

(a) *the Slater condition** *on D if there exists a point $x_0 \in D$ such that $F(x_0) < 0$;*
(b) *the weak Slater condition on D if there exists a point $x_0 \in D$ such that $F_0(x_0) < 0$, $I(x_0)$ is finite and f_i is affine for each $i \in I(x_0)$;*
(c) *the C-quasi Slater condition on D if there exists a point $x_0 \in D$ such that $F_0(x_0) < 0$, f_i is affine for each $i \in I(x_0)$ and the following closure condition holds:*

$$\mathrm{cone}\{\partial f_i(x_0) : i \in I(x_0)\} + (\mathrm{span}(C - x_0))^{\perp} \textit{ is closed;} \tag{3.7}$$

(d) *the quasi Slater condition on D if it satisfies the $\mathbb{R}^n$-quasi Slater condition on D.*

A point x_0 with the property in (a) (resp. (b), (c), (d)) *is called a Slater* (resp. *weak Slater, C-quasi Slater, quasi Slater*) *point of the system $\{f_i : i \in I\}$.*

Remark 3.1. (a) *The Slater condition (and the weak Slater condition) is one of the most important qualifications for convex systems, and has been extensively used in constrained optimization; see e.g.* [17, 28, 33]. *While the concept of the C-quasi Slater condition is new in the literature, as far as we know.*

(b) *Clearly, the Slater condition on D implies the weak Slater condition on D. We further have that the weak Slater condition on D implies the C-quasi Slater condition on D. In fact, if $x_0 \in D$ is a weak Slater point of the system $\{f_i : i \in I\}$, then $I(x_0)$ is finite and $\mathrm{cone}\{\partial f_i(x_0) : i \in I(x_0)\} + (\mathrm{span}(C - x_0))^{\perp}$ is finitely generated (noting that f_i is affine for each $i \in I(x_0)$ and that $(\mathrm{span}(C - x_0))^{\perp}$ is a subspace); thus the closure condition* (3.7) *holds.*

(c) *In the special case when $C = \mathbb{R}^n$, the closure condition* (3.7) *is reduced to the following one*

$$\mathrm{cone}\{\partial f_i(x_0) : i \in I(x_0)\} \textit{ is closed,} \tag{3.8}$$

which was introduced in [17], *where it is shown that this property is satisfied by specific families of infinite systems (like Farkas-Minkowski and locally polyhedral systems) which have a nice behavior with respect to optimality/duality theory. Hence the concept of the C-quasi Slater condition is in the spirit of the closure condition* (3.8).

*This condition is also referred as the "strong Slater condition" in some literature (e.g. [17]).

Recall that A, defined by (3.2), is the solution set of system (1.1). Following [7], the characteristic cone K of (1.1) is defined by

$$K := \text{cone}\{(\text{epi}\delta_C^*) \cup (\cup_{i\in I}\text{epi} f_i^*)\}.$$

Taking into account that $\text{epi}\delta_C^*$ is a convex cone, one has that

$$K = \text{epi}\delta_C^* + \text{cone}\{\cup_{i\in I}\text{epi} f_i^*\} = \text{epi}\delta_C^* + \sum_{i\in I}\text{cone}(\text{epi} f_i^*). \tag{3.9}$$

Note that, in our setting, the involved functions are continuous and the convex set is closed. Note also that $A \neq \emptyset$ (and so $S \neq \emptyset$) by assumption. It follows from [31, Proposition 4.1] (applied to the systems $\{\delta_C, f_i : i \in I\}$ and $\{f_i : i \in I\}$) that

$$\text{epi}\delta_A^* = \text{cl}K \quad \text{and} \quad \text{epi}\delta_S^* = \text{cl}\left(\sum_{i\in I}\text{cone}(\text{epi} f_i^*)\right); \tag{3.10}$$

see also [7, (3.1)]. In particular, if $C = \mathbb{R}^n$, then one has that $\text{epi}\delta_C^* = \{0\} \times \mathbb{R}_+$ and

$$\text{cl}\left(\sum_{i\in I}\text{cone}(\text{epi}\, f_i^*) + \{0\} \times \mathbb{R}_+\right) = \text{cl}\left(\sum_{i\in I}\text{cone}(\text{epi}\, f_i^*)\right), \tag{3.11}$$

(noting that $A = S$).

The Farkas-Minkowski (FM) type qualification was introduced in [4], which plays an important part in linear semi-infinite programming, and has been extensively developed by many authors in establishing optimality conditions, duality and stability for semi-infinite convex programming problems; see e.g. [6, 8, 15, 17, 24]. The definition is described as follows.

Definition 3.2. *The system $\{C; f_i : i \in I\}$ is Farkas-Minkowski (FM) if K is closed.*

By (3.10), the system $\{C; f_i : i \in I\}$ is FM if and only if

$$\text{epi}\delta_A^* = \text{epi}\delta_C^* + \sum_{i\in I}\text{cone}(\text{epi} f_i^*)(= K). \tag{3.12}$$

This condition (called the conical EHP therein) was introduced by Li et al. [31] to treat the case when the involved functions are not necessarily lsc and C is the whole space. Noting that the set on the right-hand side is contained in the one on the left-hand side, one has the following equivalences for the system $\{C; f_i : i \in I\}$:

$$\text{FM} \Longleftrightarrow \text{epi}\delta_A^* = K \Longleftrightarrow \text{epi}\delta_A^* \subseteq K. \tag{3.13}$$

The following proposition shows that the FM is parallel-invariant.

Proposition 3.3. *Let $x_0 \in C$ and consider the system $\{\hat{C}; \hat{f}_i : i \in I\}$ defined by*

$$\hat{C} := C - x_0 \quad \textit{and} \quad \hat{f}_i(\cdot) := f_i(\cdot + x_0) \quad \textit{for each } i \in I.$$

Then $\{C; f_i : i \in I\}$ is FM if and only if $\{\hat{C}; \hat{f}_i : i \in I\}$ is FM.

Proof. By definition, one checks that

$$\text{epi}\delta_{\hat{C}}^* = \{(x^*, \alpha) \in \mathbb{R}^n \times \mathbb{R} : (x^*, \alpha + \langle x^*, x_0\rangle) \in \text{epi}\delta_C^*\},$$

and

$$\operatorname{epi}\hat{f}_i^* = \{(x^*, \alpha) \in \mathbb{R}^n \times \mathbb{R} : (x^*, \alpha + \langle x^*, x_0\rangle) \in \operatorname{epi} f_i^*\} \quad \text{for each } i \in I.$$

Thus, it is trivial to check by (3.9) that K is closed if and only if so is the set $\hat{K} := \operatorname{epi}\delta^*_{\hat{C}} + \sum_{i\in I} \operatorname{cone}(\operatorname{epi}\hat{f}_i^*)$. This completes the proof. □

Consider a metric space I (with the metric d). Recall that a function $h : I \mapsto \overline{\mathbb{R}}$ is upper semicontinuous at $i_0 \in I$ if for any $\epsilon > 0$, there exists some $\delta > 0$ such that

$$h(i) < h(i_0) + \epsilon \quad \text{for all } i \text{ with } \mathrm{d}(i, i_0) \le \delta,$$

and that h is upper semicontinuous on I if it is upper semicontinuous at each $i \in I$. It is clear that h is upper semicontinuous at i_0 if and only if

$$\overline{\lim_{k\to\infty}} h(i_k) \le h(i_0) \quad \text{for any sequence } \{i_k\} \subseteq I \text{ with } i_k \to i_0. \tag{3.14}$$

We first have the following useful lemma.

Lemma 3.4. *Consider the system $\{f_i : i \in I\}$ of convex functions on $\mathbb{R}^n$, where I is a metric space. Suppose that, for each $x \in \mathbb{R}^n$, the function $i \mapsto f_i(x)$ is upper semicontinuous on I. Then the set-valued mapping $i \mapsto \operatorname{epi} f_i^*$ is closed in the sense that, for any sequence $\{i_k\} \subseteq I$, the relations $i_k \to i_0$ for some $i_0 \in I$ and $(h_{i_k}, \eta_{i_k}) \to (h_0, \eta_0)$ with each $(h_{i_k}, \eta_{i_k}) \in \operatorname{epi} f^*_{i_k}$ imply that $(h_0, \eta_0) \in \operatorname{epi} f^*_{i_0}$.*

Proof. Let $\{i_k\} \subseteq I$ and $\{(h_{i_k}, \eta_{i_k})\} \subset \mathbb{R}^n \times \mathbb{R}$ be sequences with each $(h_{i_k}, \eta_{i_k}) \in \operatorname{epi} f^*_{i_k}$ such that $i_k \to i_0$ and $(h_{i_k}, \eta_{i_k}) \to (h_0, \eta_0)$ for some $i_0 \in I$ and $(h_0, \eta_0) \in \mathbb{R}^n \times \mathbb{R}$. Fix $x \in \mathbb{R}^n$. Then

$$\langle h_{i_k}, x\rangle - f_{i_k}(x) \le \eta_{i_k} \quad \text{for each } k \in \mathbb{N}.$$

Passing to the limit and applying the upper semicontinuity assumption (cf. (3.14)), we obtain that

$$\langle h_0, x\rangle - f_{i_0}(x) \le \langle h_0, x\rangle - \overline{\lim_{k\to\infty}} f_{i_k}(x) \le \eta_0.$$

This shows that $(h_0, \eta_0) \in \operatorname{epi} f^*_{i_0}$ since $x \in \mathbb{R}^n$ is arbitrary, and the proof is complete. □

Let Z be a linear subspace of $\mathbb{R}^n$. For a proper function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, we use $h|_Z$ to denote the restriction of h on Z. For a proper function $h : Z \to \overline{\mathbb{R}}$ and a subset $D \subseteq Z$, to avoid confusion, we denote the epigraph of the conjugate function h^* on Z and interior of D relative to Z by $\operatorname{epi}_Z h^*$ and $\operatorname{int}_Z D$, respectively.

Lemma 3.5. *Let Z be a linear subspace of $\mathbb{R}^n$ such that $C \subseteq Z$ and suppose that the system $\{C; f_i|_Z : i \in I\}$ on Z is FM. Then the system $\{C; f_i : i \in I\}$ on $\mathbb{R}^n$ is also FM.*

Proof. Recall that A is the solution set of system (1.1). Then, by assumption ($C \subseteq Z$), we see that A coincides with the solution set of the convex system $\{C; f_i|_Z : i \in I\}$ on Z, that is,

$$A = \{z \in C : (f_i|_Z)(z) \le 0, \forall\ i\ \in I\}.$$

By (3.13), we only need to verify that $\operatorname{epi}\delta^*_A \subseteq K$. To prove this, let $(x^*, \alpha) \in \operatorname{epi}\delta^*_A$. Then $(x^*|_Z, \alpha) \in \operatorname{epi}_Z\delta^*_A$. By the assumption that $\{C; f_i|_Z : i \in I\}$ is FM, it follows from (3.13) (applied to the system $\{C; f_i|_Z : i \in I\}$ on Z in place of $\{C; f_i : i \in I\}$) that

$$\operatorname{epi}_Z\delta^*_A = \operatorname{epi}_Z\delta^*_C + \sum_{i\in I} \operatorname{cone}(\operatorname{epi}_Z(f_i|_Z)^*).$$

Therefore, there exist a finite set $J \subseteq I$, $\{\lambda_j\}_{j\in J} \subseteq \mathbb{R}_+$, $\{(\tilde{x}_j^*, \alpha_j)\}_{j\in J} \subseteq Z^* \times \mathbb{R}$ and $(\tilde{v}^*, \gamma) \in Z^* \times \mathbb{R}$, with

$$(\tilde{v}^*, \gamma) \in \mathrm{epi}_Z \delta_C^* \quad \text{and} \quad (\tilde{x}_j^*, \alpha_j) \in \mathrm{epi}_Z (f_j|_Z)^* \quad \text{for each } j \in J,$$

such that

$$(x^*|_Z, \alpha) = (\tilde{v}^*, \gamma) + \sum_{j\in J} \lambda_j (\tilde{x}_j^*, \alpha_j). \tag{3.15}$$

Fix $j \in J$ and let $x_j^* \in \mathbb{R}^n$ be an extension of $\tilde{x}_j^*$ to $\mathbb{R}^n$. Then one checks by definition that

$$(x_j^*, \alpha_j) \in \mathrm{epi}(f_j + \delta_Z)^* = \mathrm{epi}\, f_j^* + \mathrm{epi}\, \delta_Z^* = \mathrm{epi}\, f_j^* + Z^\perp \times \mathbb{R}_+, \tag{3.16}$$

where the first equality holds due to (2.4). Similarly, letting $v^* \in \mathbb{R}^n$ be an extension of $\tilde{v}^*$ to $\mathbb{R}^n$, one has that $(v^*, \gamma) \in \mathrm{epi}\delta_C^*$. Write

$$\hat{x}^* := x^* - v^* - \sum_{j\in J} \lambda_j x_j^*.$$

Then $\hat{x}^* \in Z^\perp$ due to (3.15), and thus

$$(x^*, \alpha) = (\hat{x}^*, 0) + (v^*, \gamma) + \sum_{j\in J} \lambda_j (x_j^*, \alpha_j) \in Z^\perp \times \mathbb{R}_+ + \mathrm{epi}\delta_C^* + \sum_{j\in J} \mathrm{cone}(\mathrm{epi}\, f_j^*),$$

which follows from (3.16). Noting that $Z^\perp \times \mathbb{R}_+$ is clearly contained in $\mathrm{epi}\delta_C^*$, we have that

$$(x^*, \alpha) \in Z^\perp \times \mathbb{R}_+ + \mathrm{epi}\delta_C^* + \sum_{j\in J} \mathrm{cone}(\mathrm{epi}\, f_j^*) \subseteq \mathrm{epi}\, \delta_C^* + \sum_{i\in I} \mathrm{cone}(\mathrm{epi}\, f_i^*) = K.$$

Hence the inclusion $\mathrm{epi}\delta_A^* \subseteq K$ is proved since $(x^*, \alpha) \in \mathrm{epi}\delta_A^*$ is arbitrary, and the proof is complete. □

Let D be a closed convex subset of $\mathbb{R}^n$. We say the system $\{f_i : i \in I\}$ satisfies **USC** on D if the following condition holds for each $x \in D$:

USC: I is a compact metric space, and the function $i \mapsto f_i(x)$ is upper semicontinuous on I.

It immediately follows that the assumption of the **USC** on D implies that the set $\{i \in I : f_i(x) = F(x)\}$ is non-empty and compact for each $x \in D$.

The first main theorem in this section is presented as follows, which shows that the Slater condition implies the FM qualification under the **USC** assumption of the system $\{f_i : i \in I\}$. This result is the basis for proving Theorem 3.10.

THEOREM 3.6. *Suppose that the system $\{f_i : i \in I\}$ satisfies* **USC** *on* $\mathrm{aff}C$ *and the Slater condition on C. Then the system $\{C; f_i : i \in I\}$ is FM.*

Proof. By assumption, we choose a Slater point $x_0 \in C$ for the system $\{f_i : i \in I\}$. Define

$$\hat{C} := C - x_0, \quad \hat{f}_i(\cdot) := f_i(\cdot + x_0) \quad \text{for each } i \in I.$$

Then one can easily check by definition that the system $\{\hat{f}_i : i \in I\}$ satisfies **USC** on $\mathrm{aff}\hat{C}$ and the Slater condition on $\hat{C}$. Therefore, by Proposition 3.3, we may assume, without loss of generality, that $x_0 = 0$.

Denote $Z := \operatorname{span}(C - x_0)$ ($= \operatorname{span} C$). Then, by Lemma 3.5, it suffices to prove that the new system $\{C; f_i|_Z : i \in I\}$ on Z is FM, that is,

$$\operatorname{epi}_Z \delta_A^* = \operatorname{epi}_Z \delta_C^* + \sum_{i \in I} \operatorname{cone}(\operatorname{epi}_Z (f_i|_Z)^*). \tag{3.17}$$

Note that $x_0 = 0$ is also a Slater point of the new system $\{f_i|_Z : i \in I\}$. Let

$$\tilde{S} := \{z \in Z : f_i|_Z(z) \le 0, \forall\, i \in I\} = S \cap Z.$$

Then $A = \tilde{S} \cap C$ and $x_0 \in C \cap \operatorname{int}_Z \tilde{S}$ because F is continuous and $F(x_0) < 0$. Thus, we apply Lemma 2.1 (cf. (2.4)) on Z to conclude that

$$\operatorname{epi}_Z \delta_A^* = \operatorname{epi}_Z (\delta_C + \delta_{\tilde{S}})^* = \operatorname{epi}_Z \delta_C^* + \operatorname{epi}_Z \delta_{\tilde{S}}^*.$$

To establish (3.17), it suffices to show that

$$\operatorname{epi}_Z \delta_{\tilde{S}}^* \subseteq \sum_{i \in I} \operatorname{cone}(\operatorname{epi}_Z (f_i|_Z)^*) + \operatorname{epi}_Z \delta_C^*,$$

or equivalently (by (3.10)),

$$\operatorname{cl}\left(\sum_{i \in I} \operatorname{cone}(\operatorname{epi}_Z (f_i|_Z)^*) \right) \subseteq \sum_{i \in I} \operatorname{cone}(\operatorname{epi}_Z (f_i|_Z)^*) + \operatorname{epi}_Z \delta_C^*. \tag{3.18}$$

To prove this, let $(\mu, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ and a sequence $\{(\mu_k, \alpha_k)\} \subseteq \sum_{i \in I} \operatorname{cone}(\operatorname{epi}_Z (f_i|_Z)^*)$ be such that

$$(\mu_k, \alpha_k) \to (\mu, \alpha). \tag{3.19}$$

We need to verify that

$$(\mu, \alpha) \in \sum_{i \in I} \operatorname{cone}(\operatorname{epi}_Z (f_i|_Z)^*) + \operatorname{epi}_Z \delta_C^*. \tag{3.20}$$

Fix an index $k \in \mathbb{N}$. By [36, Corollary 17.1.2], (μ_k, α_k) can be represented as

$$(\mu_k, \alpha_k) := \sum_{j \in J^k} (\mu_j, \alpha_j), \tag{3.21}$$

where $J^k := \{j_1^k, \dots, j_{n+1}^k\} \subseteq I$,

$$\mu_{j_i^k} := \lambda_{j_i^k} h_{j_i^k} \quad \text{and} \quad \alpha_{j_i^k} := \lambda_{j_i^k} \eta_{j_i^k}, \tag{3.22}$$

$$(h_{j_i^k}, \eta_{j_i^k}) \in \operatorname{epi}_Z (f_{j_i^k}|_Z)^* \quad \text{and} \quad \lambda_{j_i^k} \ge 0 \quad \text{for each } i = 1, 2, \dots, n+1.$$

Set $J = \{1, 2, \cdots, n+1\}$ and fix $i \in J$. Then

$$\alpha_{j_i^k} \ge \lambda_{j_i^k} (f_{j_i^k}|_Z)^* (h_{j_i^k}) \ge \lambda_{j_i^k} (F|_Z)^* (h_{j_i^k}), \tag{3.23}$$

where F is the sup-function defined by (3.1). This, together with the definition of $(F|_Z)^*(h_{j_i^k})$, implies that

$$\alpha_{j_i^k} \ge \lambda_{j_i^k} (\langle h_{j_i^k}, x_0 \rangle - F|_Z(x_0)) = -\lambda_{j_i^k} F|_Z(x_0) \ge 0. \tag{3.24}$$

Since by (3.19) and (3.21),

$$\lim_{k\to\infty}\sum_{i\in J}\alpha_{j_i^k} = \lim_{k\to\infty}\alpha_k = \alpha, \tag{3.25}$$

it follows that there exists $\bar{\alpha} > 0$ such that

$$0 \le \alpha_{j_i^k} \le \bar{\alpha} \quad \text{for any pair } (i,k) \in J \times \mathbb{N}. \tag{3.26}$$

Denote $\Lambda_k := \sum_{i=1}^{n+1}\lambda_{j_i^k}$. Then, we conclude from (3.24) that $-F|_Z(x_0)\Lambda_k \le \sum_{i=1}^{n+1}\alpha_{j_i^k}$. Noting that $-F|_Z(x_0) > 0$ and using (3.25), we have that $\{\Lambda_k\}$ is bounded; hence

$$\text{the sequence } \{\lambda_{j_i^k}\} \text{ is bounded for each } i \in J. \tag{3.27}$$

We set $J_1 := \{i \in J : \{h_{j_i^k}\} \text{ is bounded}\}$ and $J_2 := J \setminus J_1$. Then, by (3.22) and (3.27), we have that

$$\text{the sequence } \{\mu_{j_i^k}\} \text{ is bounded for each } i \in J_1,$$

Below we show that

$$\lim_{k\to\infty}\mu_{j_i^k} = 0 \text{ for each } i \in J_2. \tag{3.28}$$

To do this, let $i \in J_2$. Then the sequence $\{h_{j_i^k}\}$ is unbounded. By (3.22) and (3.23), we conclude that

$$\alpha_{j_i^k} \ge \lambda_{j_i^k}(F|_Z)^*(h_{j_i^k}) = \frac{\|\mu_{j_i^k}\|}{\|h_{j_i^k}\|}(F|_Z)^*(h_{j_i^k}). \tag{3.29}$$

Without loss of generality, we may assume that $\lim_{k\to\infty}\|h_{j_i^k}\| = \infty$. By assumption, F is finite on $\mathbb{R}^n$, and so $F|_Z$ is finite on Z (also by the **USC** assumption on Z). It follows from [19, p220, Proposition 1.3.9] that $(F|_Z)^*$ is 1-coercive, that is, $\lim_{\|x\|\to\infty}\frac{(F|_Z)^*(x)}{\|x\|} = +\infty$. This, together with (3.26) and (3.29), implies that $\lim_{k\to\infty}\mu_{j_i^k} = 0$, and assertion (3.28) is proved. Thus, without loss of generality, we assume that

$$j_i^k \to j_i, \quad \mu_{j_i^k} \to \mu_i, \quad \alpha_{j_i^k} \to \alpha_i, \quad \lambda_{j_i^k} \to \lambda_i \quad \text{for any } i \in J, \tag{3.30}$$

and

$$h_{j_i^k} \to h_i \quad \text{for any } i \in J_1. \tag{3.31}$$

Then, by (3.22), (3.26) and (3.28), we have

$$\alpha_i \ge 0, \quad \lambda_i \ge 0 \quad \text{for any } i \in J, \tag{3.32}$$

and

$$\mu_i = \begin{cases} \lambda_i h_i, & i \in J_1. \\ 0, & i \in J_2. \end{cases} \tag{3.33}$$

Furthermore, denoting $J_0 := \{i \in J_1 :\ \lambda_i > 0\}$, we have that

$$\eta_{j_i^k} \to \eta_i := \frac{\alpha_i}{\lambda_i} \quad \text{for any } i \in J_0. \tag{3.34}$$

By (3.21) and (3.22), we obtain

$$(\mu_k, \alpha_k) = \sum_{i\in J_0} \lambda_{j_i^k}(h_{j_i^k}, \eta_{j_i^k}) + \sum_{i\notin J_0} (\mu_{j_i^k}, \alpha_{j_i^k}).$$

Setting $\hat{\alpha} := \sum\limits_{i\notin J_0} \alpha_i \geq 0$ (see (3.32)) and taking the limit, we conclude from (3.30)-(3.34), together with (3.19), that

$$(\mu, \alpha) = \sum_{i\in J_0} \lambda_i(h_i, \eta_i) + (0, \hat{\alpha}) \tag{3.35}$$

(noting that $\mu_i = 0$ for any $i \notin J_0$ by (3.33)). By the assumption that the system $\{f_i : i \in I\}$ satisfies **USC** on Z, Lemma 3.4 is applicable to concluding that $(h_i, \eta_i) \in \mathrm{epi}_Z(f_{j_i}|_Z)^*$ for each $i \in J_0$. Thus (3.35) implies that

$$(\mu, \alpha) \in \sum_{i\in I} \mathrm{cone}(\mathrm{epi}_Z(f_i|_Z)^*) + \{0\} \times \mathbb{R}_+ \subseteq \sum_{i\in I} \mathrm{cone}(\mathrm{epi}_Z(f_i|_Z)^*) + \mathrm{epi}_Z \delta_C^*.$$

Therefore, conclusion (3.20) holds and (3.18) is shown. The proof is complete. □

In particular, when I is a finite index set, the assumption that $\{f_i : i \in I\}$ satisfies **USC** on aff$C$ holds automatically. Thus, we obtain the following corollary, which extends [21, Propositions 6.1 and 6.2] for the special case when $I := \{0\}$ is a singleton and $C := \mathbb{R}^n$.

Corollary 3.7. *Suppose that the finite system $\{f_i : i = 1, \ldots, m\}$ satisfies the Slater condition on C. Then the system $\{C; f_i : i \in I\}$ is FM.*

Let Z be a linear subspace of $\mathbb{R}^n$, and let P_Z denote the projection operator on Z. Then the operator P_Z is linear. We do not know whether the following lemma is known or not; hence we provide the proof here for the sake of completeness.

Lemma 3.8. *Let Z be a linear subspace of $\mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ be an arbitrary set. Suppose that $D + Z^\perp$ is closed. Then $P_Z(D) := \{P_Z(x) : x \in D\}$ is also closed.*

Proof. Let $z_0 \in \mathbb{R}^n$ and $\{z_k\} \subseteq P_Z(D)$ be such that

$$z_k \to z_0. \tag{3.36}$$

Then there exists a sequence $\{x_k\} \subseteq D$ such that $z_k = P_Z(x_k)$. Denote

$$x_k = z_k + z_k^\perp \quad \text{for each } k \in \mathbb{N}, \tag{3.37}$$

where $z_k^\perp := P_{Z^\perp}(x_k)$. Note that (3.37) implies that $\{z_k\} \subseteq D + Z^\perp$, and then (3.36) and the assumed closeness of $D + Z^\perp$ imply that $z_0 \in D + Z^\perp$. Thus, there exist some $x \in D$ and $y \in Z^\perp$ such that

$$z_0 = x + y. \tag{3.38}$$

Furthermore, due to $\{z_k\} \subseteq P_Z(D) \subseteq Z$, (3.36) implies that $z_0 \in Z$, and by (3.38) one has $z_0 = P_Z(z_0) = P_Z(x)$. Thus, we prove that $z_0 \in P_Z(D)$ and the proof is complete. □

Proposition 3.9. *Let $\{f_i : i \in I\}$ be a linear system (i.e., each f_i is affine). Suppose that there exists some $x_0 \in \mathrm{ri}C$ such that $f_i(x_0) = 0$ for each $i \in I$ and (3.7) holds. Then the system $\{C; f_i : i \in I\}$ is FM.*

Proof. As in the proof of Theorem 3.6, we assume, without loss of generality, that $x_0 = 0$. Then, $f_i(0) = 0$ for each $i \in I$, and thus each f_i can be expressed as $f_i(\cdot) = \langle \boldsymbol{a_i}, \cdot \rangle$, where each $\boldsymbol{a_i} \in \mathbb{R}^n$. Clearly, $\partial f_i(x_0) = \{\boldsymbol{a_i}\}$ for each $i \in I$. Let $D := \text{cone}\{\boldsymbol{a_i} : i \in I\}$, $Z := \text{span}(C - x_0)$ ($= \text{span}C$) and $\tilde{\boldsymbol{a}}_i := P_Z(\boldsymbol{a_i})$ for each $i \in I$. Then (3.7) reduces to the condition that $D + Z^\perp$ is closed. Since further the projection operator is linear, then $\text{cone}\{\tilde{\boldsymbol{a}}_{\boldsymbol{i}} : i \in I\} = P_Z(D)$ and it follows from Lemma 3.8 that

$$\text{cone}\{\tilde{\boldsymbol{a}}_{\boldsymbol{i}} : i \in I\} \text{ is closed.} \tag{3.39}$$

Moreover, it is easy to verify that

$$f_i|_Z(\cdot) = \langle \tilde{\boldsymbol{a}}_i, \cdot \rangle \quad \text{for each } i \in I,$$

and that

$$\sum_{i \in I} \text{cone}(\text{epi}_Z(f_i|_Z)^*) + \{0\} \times \mathbb{R}_+ = \text{cone}\{\tilde{\boldsymbol{a}}_{\boldsymbol{i}} : i \in I\} \times \mathbb{R}_+.$$

This, together with (3.39), implies that the system $\{Z, f_i|_Z : i \in I\}$ is FM (on Z). Hence, by (3.13), we have

$$\text{epi}_Z \delta^*_{\tilde{S}} = \sum_{i \in I} \text{cone}(\text{epi}_Z(f_i|_Z)^*) + \{0\} \times \mathbb{R}_+, \tag{3.40}$$

where $\tilde{S} := \{z \in Z : f_i|_Z(z) \le 0, \forall\, i \in I\} = S \cap Z$. By the assumptions that $A = C \cap \tilde{S}$ and $x_0 \in \tilde{S} \cap \text{int}_Z C$ (due to $x_0 \in \text{ri}C$ and $Z = \text{span}(C - x_0)$), (2.4) implies that

$$\text{epi}_Z \delta^*_A = \text{epi}_Z(\delta_{\tilde{S}} + \delta_C)^* = \text{epi}_Z \delta^*_C + \text{epi}_Z \delta^*_{\tilde{S}}.$$

Thus, by (3.40), the following equality holds:

$$\text{epi}_Z \delta^*_A = \text{epi}_Z \delta^*_C + \sum_{i \in I} \text{cone}(\text{epi}_Z(f_i|_Z)^*) + \{0\} \times \mathbb{R}_+ = \text{epi}_Z \delta^*_C + \sum_{i \in I} \text{cone}(\text{epi}_Z(f_i|_Z)^*),$$

and it follows from (3.13) that the system $\{C; f_i|_Z : i \in I\}$ on Z is FM. Consequently, the system $\{C; f_i : i \in I\}$ is FM due to Lemma 3.5, and the proof is complete. □

Extending Theorem 3.6, the following theorem demonstrates that the C-quasi Slater condition implies the FM qualification under the **USC** assumption of the system $\{f_i : i \in I\}$.

THEOREM 3.10. *Suppose that the system $\{f_i : i \in I\}$ satisfies* **USC** *on* $\text{aff}\, C$ *and the C-quasi Slater condition on* $\text{ri}\, C$. *Then the system $\{C; f_i : i \in I\}$ is FM.*

Proof. Let $x_0 \in \text{ri}C$ be a C-quasi Slater point of the system $\{f_i : i \in I\}$. Recall that $I(x_0)$ is the active index set at x_0 defined by (3.4) and $I_0 = I \setminus I(x_0)$ is defined by (3.5). We assume, without loss of generality, that $I(x_0) \neq \emptyset$ (otherwise, the C-quasi Slater condition is reduced to the Slater condition). Denote $\tilde{I} := \text{cl}(I_0)$. Then $\tilde{I} \subseteq I$ is also a compact metric space. For each $i \in \tilde{I}$, define $\tilde{f}_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$\tilde{f}_i(x) := \sup \left\{ \overline{\lim_k} f_{j_k}(x) : \text{ sequence } \{j_k\} \subseteq I_0, j_k \to i \right\} \quad \text{for each } x \in \mathbb{R}^n \tag{3.41}$$

if $i \in \tilde{I} \cap I(x_0)$; and $\tilde{f}_i := f_i$, otherwise. Clearly, $\tilde{f}_i$ is convex and finite on $\mathbb{R}^n$, because the sup-function F is finite. Consider the new system $\{C; \tilde{f}_i : i \in \tilde{I}\}$ with its solution set denoted by $\tilde{A}_0$. Below, we show the following assertions:

(a) For any $i \in \tilde{I}$, we have $\tilde{f}_i(x) \le f_i(x)$ for each $x \in \mathrm{aff}C$.
(b) The system $\{\tilde{f}_i : i \in \tilde{I}\}$ satisfies **USC** on aff$C$.
(c) The system $\{\tilde{f}_i : i \in \tilde{I}\}$ satisfies the Slater condition on C.

Indeed, assertion (a) is a direct consequence of the definition of system $\{\tilde{f}_i : i \in \tilde{I}\}$ and the assumption that the system $\{f_i : i \in I\}$ satisfies **USC** on aff$C$. To show assertion (b), let $x \in \mathrm{aff}C$, $i_0 \in \tilde{I}$ and a sequence $\{j_k\} \subseteq \tilde{I}$ be such that $j_k \to i_0$. We only need to prove that

$$\overline{\lim_k} \tilde{f}_{j_k}(x) \le \tilde{f}_{i_0}(x) \tag{3.42}$$

(by the equivalent condition (3.14) of upper semicontinuity). If $i_0 \in I_0$, we have that $\tilde{f}_{i_0}(x) = f_{i_0}(x)$ by definition; hence assertion (a) and the assumption that the system $\{f_i : i \in I\}$ satisfies **USC** on aff$C$ imply that

$$\overline{\lim_k} \tilde{f}_{j_k}(x) \le \overline{\lim_k} f_{j_k}(x) \le f_{i_0}(x) = \tilde{f}_{i_0}(x),$$

that is, (3.42) is proved. Now, we consider the case when $i_0 \in \tilde{I} \cap I(x_0)$. Without loss of generality, we assume that $\lim_k \tilde{f}_{j_k}(x)$ exists and

$$\overline{\lim_k} \tilde{f}_{j_k}(x) = \lim_k \tilde{f}_{j_k}(x) \tag{3.43}$$

(otherwise, one can choose a subsequence satisfying (3.43)). Furthermore, we may assume that $\{j_k\} \subseteq \tilde{I} \cap I(x_0)$ (otherwise, $\{j_k\}$ contains a subsequence, denoted by itself, such that $\{j_k\} \subseteq I_0$, and then $\lim_k \tilde{f}_{j_k}(x) = \lim_k f_{j_k}(x) \le \tilde{f}_{i_0}(x)$ by (3.41)). For each $k \in \mathbb{N}$, since $j_k \in \tilde{I} \cap I(x_0)$, it follows from (3.41) that there exists some $i_k \in I_0$ such that

$$f_{i_k}(x) \ge \tilde{f}_{j_k}(x) - \frac{1}{k} \quad \text{and} \quad \mathrm{d}(i_k, j_k) \le \frac{1}{k}.$$

Noting that $j_k \to i_0$, one checks that $i_k \to i_0$ and

$$\lim_k \tilde{f}_{j_k}(x) \le \overline{\lim_k} f_{i_k}(x) \le \tilde{f}_{i_0}(x).$$

Thus (3.42) holds (noting (3.43)), and assertion (b) is established. To check assertion (c), noting that $\tilde{I} = \mathrm{cl}(I_0)$ and (3.41), one obtains that

$$\tilde{F}(x) := \sup_{i \in \tilde{I}} \tilde{f}_i(x) = \sup_{i \in I_0} \tilde{f}_i(x) = \sup_{i \in I_0} f_i(x) \quad \text{for each } x \in \mathrm{aff}C. \tag{3.44}$$

Since $x_0 \in \mathrm{ri}C$ is a C-quasi Slater point of the system $\{f_i : i \in I\}$, one has that $F_0(x_0) < 0$ (by (3.6)). This, together with (3.44), implies

$$\tilde{F}(x_0) = \sup_{i \in \tilde{I}} \tilde{f}_i(x_0) = \sup_{i \in I_0} f_i(x_0) < 0.$$

Hence x_0 is a Slater point of the system $\{\tilde{f}_i : i \in \tilde{I}\}$, and assertion (c) is then shown. Therefore, applying Theorem 3.6 to the system $\{\tilde{f}_i : i \in \tilde{I}\}$, one has that

$$\mathrm{epi}\delta^*_{\tilde{A}_0} = \mathrm{epi}\delta^*_C + \sum_{i \in \tilde{I}} \mathrm{cone}(\mathrm{epi}\tilde{f}_i^*). \tag{3.45}$$

Denote

$$S_0 := \{x \in \mathbb{R}^n : f_i(x) \leq 0, \forall\, i \in I_0\} \quad \text{and} \quad \tilde{S}_0 := \{x \in \mathbb{R}^n : \tilde{f}_i(x) \leq 0, \forall\, i \in \tilde{I}\}. \tag{3.46}$$

Then $\tilde{A}_0 = C \cap \tilde{S}_0$, and by (3.44), we obtain

$$C \cap \tilde{S}_0 = \{x \in C : \tilde{F}(x) \leq 0\} = \{x \in C : \sup_{i \in I_0} f_i(x) \leq 0\} = C \cap S_0.$$

Hence,

$$\tilde{A}_0 = C \cap \tilde{S}_0 = C \cap S_0 \subseteq S_0. \tag{3.47}$$

Moreover, for each $i \in \tilde{I}$, assertion (a) implies that

$$\tilde{f}_i \leq \tilde{f}_i + \delta_C \leq f_i + \delta_C,$$

and it follows from (2.1) and (2.4) that

$$\mathrm{epi}\tilde{f}_i^* \subseteq \mathrm{epi}(\tilde{f}_i + \delta_C)^* \subseteq \mathrm{epi}(f_i + \delta_C)^* = \mathrm{epi}\delta_C^* + \mathrm{epi}f_i^*.$$

This, together with (2.2), (3.45) and (3.47), implies that

$$\mathrm{epi}\delta_{S_0}^* \subseteq \mathrm{epi}\delta_{\tilde{A}_0}^* \subseteq \mathrm{epi}\delta_C^* + \sum_{i \in \tilde{I}} \mathrm{cone}(\mathrm{epi}f_i^*). \tag{3.48}$$

(Note that this inclusion doesn't require the assumption that $x_0 \in \mathrm{ri}C$). Let A_1 denote the solution set of the system $\{C; f_i : i \in I(x_0)\}$, that is,

$$A_1 := \{x \in C : f_i(x) \leq 0, \forall\, i \in I(x_0)\}.$$

Then, $A = S_0 \cap A_1$. Since $S_0 = \{x \in \mathbb{R}^n : F_0(x) \leq 0\}$ by (3.46) and $F_0(x_0) < 0$ as noted earlier, we further have that $x_0 \in \mathrm{int}S_0 \cap A_1$ (as F_0 is finite on $\mathbb{R}^n$ by assumption). Thus, by (2.4), one has

$$\mathrm{epi}\delta_A^* = \mathrm{epi}(\delta_{S_0} + \delta_{A_1})^* = \mathrm{epi}\delta_{S_0}^* + \mathrm{epi}\delta_{A_1}^*. \tag{3.49}$$

Moreover, Proposition 3.9 says that the system $\{C; f_i : i \in I(x_0)\}$ is FM, that is,

$$\mathrm{epi}\delta_{A_1}^* = \mathrm{epi}\delta_C^* + \sum_{i \in I(x_0)} \mathrm{cone}(\mathrm{epi}\, f_i^*). \tag{3.50}$$

Combining (3.48), (3.49) and (3.50) yields that

$$\mathrm{epi}\delta_A^* \subseteq \mathrm{epi}\delta_C^* + \sum_{i \in I} \mathrm{cone}(\mathrm{epi}\, f_i^*).$$

This shows that the system $\{C; f_i : i \in I\}$ is FM, and the proof is complete. □

The proof of Theorem 3.10 actually verified the following remark, which will be useful in the next section.

Remark 3.2. *Under the assumption that the system $\{f_i : i \in I\}$ satisfies* **USC** *on* $\mathrm{aff}\, C$ *and the C-quasi Slater condition on C. Then the following inclusion holds:*

$$\mathrm{epi}\delta_{S_0}^* \subseteq \mathrm{epi}\delta_C^* + \sum_{i \in I} \mathrm{cone}(\mathrm{epi}f_i^*),$$

where $S_0 := \{x \in \mathbb{R}^n : f_i(x) \le 0, \forall\, i \in I \setminus I(x_0)\}$.

By Remark 3.1 (b), the following corollary is a direct consequence of Theorem 3.10.

COROLLARY 3.11. *Suppose that the system* $\{f_i : i \in I\}$ *satisfies* **USC** *on* aff$C$ *and the weak Slater condition on* ri$C$. *Then the system* $\{C; f_i : i \in I\}$ *is FM.*

As noted earlier, when I is finite, the **USC** assumption holds automatically and we conclude the following corollary, which further extends and improves Corollary 3.7 and [21, Propositions 6.1, 6.2].

COROLLARY 3.12. *Suppose that the finite system* $\{f_i : i = 1, \dots, m\}$ *satisfies the weak Slater condition on* ri$C$. *Then the system* $\{C; f_i : i \in I\}$ *is FM.*

Recall from [28] (also one can refer to [31]) that the system $\{f_i : i \in I\}$ satisfies the BCQ relative to C if

$$N_A(x) = N_C(x) + \text{cone} \bigcup_{i \in I(x)} \partial f_i(x) \quad \text{for each } x \in A.$$

It was proved in [28, Corollary 3.7] that if the system $\{f_i : i \in I\}$ satisfies **USC** on the whole space and the weak Slater condition on ri$C$, then it satisfies the BCQ relative to C. In the finite dimensional space, the following corollary extends this result in two folds: one is that the weak Slater condition can be weakened to the C-quasi Slater condition, and the other is that the **USC** holds only on aff$C$ rather than the whole space.

COROLLARY 3.13. *Suppose that the system* $\{f_i : i \in I\}$ *satisfies* **USC** *on* aff$C$ *and the* C*-quasi Slater condition on* ri$C$. *Then the system* $\{f_i : i \in I\}$ *satisfies the BCQ relative to* C.

Proof. By Theorem 3.10, the system $\{C; f_i : i \in I\}$ is FM. Noting (3.12), it is equivalent to that the system $\{\delta_C; f_i : i \in I\}$ satisfies the conical EHP. Then, it follows from [31, Theorem 4.1 (ii)] that the system $\{f_i : i \in I\}$ satisfies the BCQ relative to C and the proof is complete. □

We end this section with two examples: the first one provides a convex system that has a C-quasi Slater point but does not have any weak Slater point, and the second one gives a convex system that satisfies **USC** on aff$C$ but not on $\mathbb{R}^n$.

EXAMPLE 3.1. Let $I := [-1, 1]$, $C = \mathbb{R}$ and the system $\{f_i : i \in I\}$ on $\mathbb{R}$ be defined by

$$f_i(x) := \begin{cases} ix, & i \in [-1, 0] \\ ix - 1, & i \in (0, 1] \end{cases} \quad \text{for each } x \in \mathbb{R}.$$

Then, $S = [0, 1]$ and the system $\{f_i : i \in I\}$ satisfies **USC** on $\mathbb{R}$, that is, the function $i \mapsto f_i(x)$ is upper semicontinuous on $\mathbb{R}$. Let $x_0 := 0$. Then $I(x_0) = [-1, 0]$, and the corresponding sup-function F_0 is given by

$$F_0(x) := \sup_{i \in I \setminus I(x_0)} f_i(x) = \begin{cases} x - 1, & x > 0 \\ -1, & x \le 0 \end{cases} \quad \text{for each } x \in \mathbb{R}.$$

This means that F_0 is continuous and $F_0(x_0) < 0$. Note further that f_i is affine and $\partial f_i(x_0) = i$ for each $i \in I$. Thus, we have that cone$\{\partial f_i(x_0) : i \in I(x_0)\} = (-\infty, 0]$ is closed. This shows that $x_0 = 0$ is a quasi Slater point of the system $\{f_i : i \in I\}$ on $\mathbb{R}$, and hence Corollary 3.13 is applicable to concluding that the system $\{f_i : i \in I\}$ satisfies the BCQ. However, $x_0 = 0$ is not a weak Slater point. Below we show that the system $\{f_i : i \in I\}$ does not satisfy the weak Slater condition on $\mathbb{R}$, and thus [28, Corollary 3.7] cannot be applied. In fact, when $x_0 \in (0,1]$,

$$I(x_0) = \begin{cases} \{0,1\}, & x_0 = 1, \\ \{0\}, & 0 < x_0 < 1. \end{cases}$$

Clearly, $F_0(x_0) = 0$. Therefore, any $x_0 \in [0,1]$ is not a weak Slater point of the system $\{f_i : i \in I\}$ and the system does not satisfy the weak Slater condition on $\mathbb{R}$.

EXAMPLE 3.2. Let $I := \{0, 1, \frac{1}{2}, \cdots, \frac{1}{k}, \cdots\}$, $C := [-2, 0] \times \{1\}$ and the system $\{f_i : i \in I\}$ on $\mathbb{R}^2$ be defined by

$$f_i(x) := \begin{cases} t_1 + t_2, & i = 0 \\ t_1 + it_2, & i \in I \backslash \{0\} \end{cases} \qquad \text{for each } x = (t_1, t_2) \in \mathbb{R}^2.$$

Then, $\mathrm{aff}C = \mathbb{R} \times \{1\}$ and the system $\{f_i : i \in I\}$ satisfies **USC** on $\mathrm{aff}C$ but not on $\mathbb{R}^2$ (e.g. at $(-1,-1)$). Thus, [28, Corollary 3.7] cannot be applied, despite we show below that the system $\{f_i : i \in I\}$ satisfies the weak Slater condition on $\mathrm{ri}C$. To show this, set $x_0 := (-1, 1)$. Then $x_0 \in \mathrm{ri}C$, $I(x_0) = \{0, 1\}$ and $F_0(x_0) < 0$. Since f_i is linear for each $i \in I$, it follows that x_0 is a weak Slater point of the system $\{f_i : i \in I\}$. Therefore, Corollary 3.13 is applicable to concluding that the system $\{f_i : i \in I\}$ satisfies the BCQ relative to C.

4. Applications. The main aim of this section is to apply our preceding results to some typical problems, including set containment characterization, duality problems and Farkas lemma.

4.1. Set containment characterization. Let $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ be two systems of functions on $\mathbb{R}^n$. The set containment problem is to seek conditions to characterize the following inclusion:

$$\{x \in \mathbb{R}^n : f_i(x) \le 0, \forall\, i \in I\} \subseteq \{x \in \mathbb{R}^n : g_j(x) \le 0, \forall\, j \in J\}.$$

Throughout this subsection, we assume that I, J are arbitrary index sets (possibly infinite), $\{f_i : i \in I\}$ is a system of finite-valued convex functions and $\{g_j : j \in J\}$ is a system of finite-valued DC functions (difference of two convex functions) on $\mathbb{R}^n$ with each g_j defined by

$$g_j := \varphi_j - \psi_j \quad \text{for each } j \in J,$$

where $\{\varphi_j : j \in J\}$ and $\{\psi_j : j \in J\}$ are systems of finite-valued convex functions on $\mathbb{R}^n$. The objective of this subsection is to establish dual characterizations for these kinds of set containment, both asymptotic and nonasymptotic types. As corollaries, we also provide dual characterizations for the above inclusion when $\{g_j : j \in J\}$ is a system of affine, convex or concave functions, respectively.

The equivalence between (i) and (ii) in the following theorem was described by Jeyakumar in [21, Theorem 4.2] only for the case when the index set J is finite. Here we extend the corresponding result into infinitely many constraints and establish the nonasymptotic dual characterization (iii) under the **USC** assumption and the quasi Slater condition.

THEOREM 4.1. *Consider the following statements:*

(i) $\{x \in \mathbb{R}^n : f_i(x) \le 0, \forall\, i \in I\} \subseteq \{x \in \mathbb{R}^n : g_j(x) \le 0, \forall\, j \in J\}$*;*

(ii) *for each* $j \in J$*,*

$$\mathrm{epi}\,\varphi_j^* \subseteq \mathrm{epi}\,\psi_j^* + \mathrm{cl}\left(\sum_{i \in I} \mathrm{cone}(\mathrm{epi}\, f_i^*)\right);$$

(iii) *for each* $j \in J$,

$$\operatorname{epi}\varphi_j^* \subseteq \operatorname{epi}\psi_j^* + \sum_{i\in I}\operatorname{cone}(\operatorname{epi} f_i^*).$$

Then (i) $\Leftrightarrow$ (ii). *In addition, if the system* $\{f_i : i \in I\}$ *satisfies* **USC** *and the quasi Slater condition on* $\mathbb{R}^n$, *then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. Recall that S is defined by (3.3) and let $H_j := \{x \in \mathbb{R}^n : \varphi_j(x) - \psi_j(x) \le 0\}$, for each $j \in J$. Then we have the following equivalences:

$$\begin{aligned}
\text{(i)} \quad &\Leftrightarrow S \subseteq H_j \quad \text{for each } j \in J,\\
&\Leftrightarrow \varphi_j \le \psi_j + \delta_S \quad \text{for each } j \in J,\\
&\Leftrightarrow \operatorname{epi}\varphi_j^* \subseteq \operatorname{epi}(\psi_j + \delta_S)^* \quad \text{for each } j \in J,
\end{aligned} \tag{4.1}$$

where the last equivalence holds due to (2.1).

On the other hand, since, for each $j \in J$, ψ_j is a finite convex function on $\mathbb{R}^n$, then it is continuous. Thus it follows from (2.4) that

$$\operatorname{epi}(\psi_j + \delta_S)^* = \operatorname{epi}\psi_j^* + \operatorname{epi}\delta_S^*. \tag{4.2}$$

Moreover, by (3.10), one has

$$\operatorname{epi}\delta_S^* = \operatorname{cl}\left(\sum_{i\in I}\operatorname{cone}(\operatorname{epi} f_i^*)\right). \tag{4.3}$$

Consequently, (4.2) and (4.3) imply that

$$\operatorname{epi}(\psi_j + \delta_S)^* = \operatorname{epi}\psi_j^* + \operatorname{cl}\left(\sum_{i\in I}\operatorname{cone}(\operatorname{epi} f_i^*)\right) \quad \text{for each } j \in J. \tag{4.4}$$

Therefore, by (4.1) and (4.4), we establish the equivalence between (i) and (ii).

In addition, if the system $\{f_i : i \in I\}$ satisfies **USC** and the quasi Slater condition on $\mathbb{R}^n$, then it follows from Theorem 3.10 that the convex system $\{\mathbb{R}^n, f_i : i \in I\}$ is FM. Then, by (3.11), we have

$$\operatorname{epi}\psi_j^* + \operatorname{cl}\left(\sum_{i\in I}\operatorname{cone}(\operatorname{epi} f_i^*)\right) = \operatorname{epi}\psi_j^* + \sum_{i\in I}\operatorname{cone}(\operatorname{epi} f_i^*) + \{0\}\times\mathbb{R}_+ = \operatorname{epi}\psi_j^* + \sum_{i\in I}\operatorname{cone}(\operatorname{epi} f_i^*).$$

Thus, we arrive at the equivalence between (ii) and (iii). The proof is complete. □

Restricted g_j to be affine, the following corollary, extending [21, Theorem 3.2 and Theorem 3.5], provides the dual characterization of the set containment of a closed convex set in the intersection of infinitely many half-spaces. The equivalence between (i) and (ii) in the following corollary was given in [21, Theorem 3.2] only when the index set J is finite. Furthermore, when J is finite, the equivalence between (i) and (iii) for the case $u_j \ne 0$ for each $j \in J$ was shown in [21, Theorem 3.5] under the Slater condition on $\mathbb{R}^n$ of the system $\{f_i : i \in I\}$.

COROLLARY 4.2. *Let* $u_j \in \mathbb{R}^n$ *and* $a_j \in \mathbb{R}$ *for each* $j \in J$. *Consider the following statements:*

(i) $\{x \in \mathbb{R}^n : f_i(x) \le 0, \forall\, i \in I\} \subseteq \{x \in \mathbb{R}^n : \langle u_j, x\rangle \le a_j, \forall\, j \in J\}$;

(ii) *for each* $j \in J$,

$$(u_j, a_j) \in \mathrm{cl}\left(\sum_{i\in I} \mathrm{cone}(\mathrm{epi}\, f_i^*)\right);$$

(iii) *for each* $j \in J$,

$$(u_j, a_j) \in \sum_{i\in I} \mathrm{cone}(\mathrm{epi}\, f_i^*) + \{0\} \times \mathbb{R}_+.$$

Then (i) $\Leftrightarrow$ (ii). *In addition, if the system* $\{f_i : i \in I\}$ *satisfies* **USC** *and the quasi Slater condition on* $\mathbb{R}^n$, *then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. For each $j \in J$, by taking $\varphi_j(\cdot) := \langle u_j, \cdot\rangle - a_j$ and $\psi_j := 0$, $\mathrm{epi}\varphi_j^* = \{u_j\} \times [a_j, +\infty)$ and $\mathrm{epi}\psi_j^* = \{0\} \times \mathbb{R}_+$. Hence by (3.11), one sees that statement (ii) of Theorem 4.1 is reduced to

$$\mathrm{epi}\, \varphi_j^* \subseteq \{0\} \times \mathbb{R}_+ + \mathrm{cl}\left(\sum_{i\in I} \mathrm{cone}(\mathrm{epi}\, f_i^*)\right) = \mathrm{cl}\left(\sum_{i\in I} \mathrm{cone}(\mathrm{epi}\, f_i^*)\right). \tag{4.5}$$

Thus (i)$\Leftrightarrow$(ii) is seen to hold by Theorem 4.1. Furthermore, the equivalence between (ii) and (iii) follows from (3.11) and Theorem 3.10. $\square$

In the special case when each g_j is concave/convex, the following corollaries present the dual characterizations of set containment of a closed convex set in a reverse-convex/convex set with infinitely many constraints. When the index set J is finite, the equivalences between (i) and (ii) were obtained in [21, Theorem 4.1] and [21, Corollary 4.3], respectively. Another improvement is that the nonasymptotic dual characterizations (iii) are given under the **USC** and quasi Slater condition.

Corollary 4.3. *Consider the following statements:*

(i) $\{x \in \mathbb{R}^n : f_i(x) \le 0, \forall\, i \in I\} \subseteq \{x \in \mathbb{R}^n : \psi_j(x) \ge 0, \forall\, j \in J\}$;

(ii) *for each* $j \in J$,

$$0 \in \mathrm{epi}\, \psi_j^* + \mathrm{cl}\left(\sum_{i\in I} \mathrm{cone}(\mathrm{epi}\, f_i^*)\right);$$

(iii) *for each* $j \in J$,

$$0 \in \mathrm{epi}\, \psi_j^* + \sum_{i\in I} \mathrm{cone}(\mathrm{epi}\, f_i^*).$$

Then (i) $\Leftrightarrow$ (ii). *In addition, if the system* $\{f_i : i \in I\}$ *satisfies* **USC** *and the quasi Slater condition on* $\mathbb{R}^n$, *then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. For each $j \in J$, by taking $\varphi_j := 0$, $\mathrm{epi}\varphi_j^* = \{0\} \times \mathbb{R}_+$. Hence by Theorem 4.1, statement (i) is equivalent to that $\{0\} \times \mathbb{R}_+ \subseteq \mathrm{epi}\, \psi_j^* + \mathrm{cl}\left(\sum_{i\in I} \mathrm{cone}(\mathrm{epi}\, f_i^*)\right)$, which by definition is equivalent to $0 \in \mathrm{epi}\, \psi_j^* + \mathrm{cl}\left(\sum_{i\in I} \mathrm{cone}(\mathrm{epi}\, f_i^*)\right)$, for each $j \in J$. Similarly, the equivalence between (ii) and (iii) follows from (3.11) and Theorem 3.10. $\square$

The proof of the following corollary uses a line of analysis similar to that of Corollary 4.3, by taking $\psi_j := 0$ for each $j \in J$.

Corollary 4.4. *Consider the following statements:*

(i) $\{x \in \mathbb{R}^n : f_i(x) \leq 0, \forall\, i \in I\} \subseteq \{x \in \mathbb{R}^n : \varphi_j(x) \leq 0, \forall\, j \in J\}$;

(ii) *for each* $j \in J$,

$$\operatorname{epi} \varphi_j^* \subseteq \operatorname{cl}\left(\sum_{i \in I} \operatorname{cone}(\operatorname{epi} f_i^*)\right);$$

(iii) *for each* $j \in J$,

$$\operatorname{epi} \varphi_j^* \subseteq \sum_{i \in I} \operatorname{cone}(\operatorname{epi} f_i^*) + \{0\} \times \mathbb{R}_+.$$

Then (i) $\Leftrightarrow$ (ii). *In addition, if the system* $\{f_i : i \in I\}$ *satisfies* **USC** *and the quasi Slater condition on* $\mathbb{R}^n$, *then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. Let $j \in J$ and $\psi_j := 0$. Then $\operatorname{epi}\psi_j^* = \{0\} \times \mathbb{R}_+$. Hence, the equivalence between (i) and (ii) follows from Theorem 4.1 and (4.5), and the equivalence between (ii) and (iii) follows from (3.11) and Theorem 3.10. □

4.2. Strong Lagrangian duality and Farkas lemma. The objective of this subsection is to provide some sufficient conditions for ensuring the strong Lagrangian duality and Farkas lemma. Following [2], we use $\mathbb{R}^{(\mathrm{I})}$ to denote the space of real tuples $\lambda = (\lambda_i)_{i \in I}$ with only finitely many $\lambda_i \neq 0$, and let $\mathbb{R}_+^{(\mathrm{I})}$ denote the nonnegative cone in $\mathbb{R}^{(\mathrm{I})}$, that is,

$$\mathbb{R}_+^{(I)} := \left\{(\lambda_i)_{i \in I} \in \mathbb{R}^{(I)} : \lambda_i \geq 0 \text{ for each } i \in I\right\}.$$

Use (P_f) to denote the problem (1.2) and recall that f is a proper convex function satisfying $\operatorname{dom} f \cap A \neq \emptyset$. Define the Lagrangian function L_f on $\mathbb{R}^n \times \mathbb{R}_+^{(\mathrm{I})}$ by

$$L_f(x, \lambda) := f(x) + \sum_{i \in I} \lambda_i f_i(x) \text{ for each } (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}_+^{(\mathrm{I})}.$$

Then the Lagrangian dual problem of (P_f) is given by

$$(D_f) \qquad \begin{array}{ll} \text{Maximize} & \inf\limits_{x \in C} L_f(x, \lambda), \\ \text{subject to} & \lambda \in \mathbb{R}_+^{(I)}. \end{array}$$

We denote by $v(P_f)$ and $v(D_f)$ the optimal objective values of (P_f) and (D_f) respectively. We say that the strong Lagrangian duality between (P_f) and (D_f) holds if $v(P_f) = v(D_f)$ and the dual problem (D_f) has an optimal solution.

Recall that the system $\{f, \delta_C; f_i : i \in I\}$ is said to satisfy the Farkas rule if, for each $\alpha \in \mathbb{R}$, it holds that

$$[f(x) \geq \alpha,\ \forall x \in A] \Longleftrightarrow \left[\exists \lambda \in \mathbb{R}_+^{(I)} \text{ s.t. } f(x) + \sum_{i \in I} \lambda_i f_i(x) \geq \alpha\ \forall x \in C\right].$$

The relations between the Lagrangian duality, the Farkas lemma and constraint qualifications were detailedly studied in [12, Theorem 4.4, Corollary 5.3]. Here we restate them as the following proposition, which will be used in the sequel.

PROPOSITION 4.5. *Suppose that the system* $\{f, \delta_C; f_i : i \in I\}$ *satisfies*

$$\operatorname{epi}(f+\delta_A)^* \subseteq \bigcup_{\lambda \in \mathbb{R}_+^{(I)}} \operatorname{epi}\left(f + \delta_C + \sum_{i \in I} \lambda_i f_i\right)^* . \tag{4.6}$$

Then the strong Lagrangian duality between (P_f) *and* (D_f) *holds and the system* $\{f, \delta_C; f_i : i \in I\}$ *satisfies the Farkas rule.*

THEOREM 4.6. *Suppose that the system* $\{f_i : i \in I\}$ *satisfies* **USC** *on* aff$C$ *and the Slater condition on* dom$f \cap C$. *Then the strong Lagrangian duality between* (P_f) *and* (D_f) *holds and the system* $\{f, \delta_C; f_i : i \in I\}$ *satisfies the Farkas rule.*

Proof. Since the system $\{f_i : i \in I\}$ satisfies the Slater condition on dom$f \cap C$, there exists a point $x_0 \in \operatorname{dom} f \cap C$ such that $F(x_0) < 0$; hence $x_0 \in \operatorname{dom} f \cap C \cap (\operatorname{int} S)$. Thus, applying (2.4) (to $f + \delta_C$ and δ_S in place of f and h), we have

$$\operatorname{epi}(f+\delta_A)^* = \operatorname{epi}(f+\delta_C+\delta_S)^* = \operatorname{epi}(f+\delta_C)^* + \operatorname{epi}\delta_S^*. \tag{4.7}$$

On the other hand, $A \subseteq S$ implies (cf. (2.2))

$$\operatorname{epi}\delta_S^* \subseteq \operatorname{epi}\delta_A^*, \tag{4.8}$$

and Theorem 3.6 implies

$$\operatorname{epi}\delta_A^* = \operatorname{epi}\delta_C^* + \textstyle\sum_{i \in I} \operatorname{cone}(\operatorname{epi} f_i^*), \tag{4.9}$$

Then, combining (4.7)-(4.9), and applying (2.3), one has

$$\begin{aligned} \operatorname{epi}(f+\delta_A)^* \quad &\subseteq \operatorname{epi}(f+\delta_C)^* + \operatorname{epi}\delta_C^* + \textstyle\sum_{i \in I} \operatorname{cone}(\operatorname{epi} f_i^*) \\ &\subseteq \cup_{\lambda \in \mathbb{R}_+^{(I)}} \operatorname{epi}\left(f + \delta_C + \textstyle\sum_{i \in I} \lambda_i f_i\right)^* . \end{aligned} \tag{4.10}$$

Thus, (4.6) in Proposition 4.5 is verified and the conclusion follows from Proposition 4.5. □

THEOREM 4.7. *Suppose that the system* $\{f_i : i \in I\}$ *satisfies* **USC** *on* aff$C$ *and the* C*-quasi Slater condition on* ri$C$. *Furthermore, suppose the following condition holds:*

$$\operatorname{epi}(f+\delta_A)^* = \operatorname{epi} f^* + \operatorname{epi}\delta_A^*. \tag{4.11}$$

Then the strong Lagrangian duality between (P_f) *and* (D_f) *holds and the system* $\{f, \delta_C; f_i : i \in I\}$ *satisfies the Farkas rule.*

Proof. By Theorem 3.10, one has

$$\operatorname{epi}\delta_A^* = \operatorname{epi}\delta_C^* + \sum_{i \in I} \operatorname{cone}(\operatorname{epi} f_i^*).$$

This, together with (4.11) and (2.3), implies that

$$\begin{aligned} \operatorname{epi}(f+\delta_A)^* \quad &= \operatorname{epi} f^* + \operatorname{epi}\delta_C^* + \textstyle\sum_{i \in I} \operatorname{cone}(\operatorname{epi} f_i^*) \\ &\subseteq \cup_{\lambda \in \mathbb{R}_+^{(\mathrm{I})}} \operatorname{epi}\left(f + \delta_C + \textstyle\sum_{i \in I} \lambda_i f_i\right)^* . \end{aligned}$$

Thus, the conclusion follows from Proposition 4.5. □

The following theorem shows that if we replace the C-quasi Slater condition in Theorem 4.7 by the weak Slater condition on $\mathrm{ri}(\mathrm{dom} f \cap C)$, the assumption (4.11) in Theorem 4.7 can be dropped. For the sake of completeness, we present the following proposition, which is a restatement of [36, Theorem 28.2] in terms of the weak Slater condition if $\inf_{x\in A} f(x) \neq -\infty$, and holds trivially otherwise.

Proposition 4.8. *Let I be finite and suppose that the system $\{f_i : i \in I\}$ satisfies the weak Slater condition on $\mathrm{ri} C$. Suppose further that $C \subseteq \mathrm{dom} f$. Then the strong Lagrangian duality between (P_f) and (D_f) holds.*

Theorem 4.9. *Suppose that the system $\{f_i : i \in I\}$ satisfies* **USC** *on $\mathrm{aff} C$ and the weak Slater condition on $\mathrm{ri}(\mathrm{dom} f \cap C)$. Then the strong Lagrangian duality between (P_f) and (D_f) holds and the system $\{f, \delta_C; f_i : i \in I\}$ satisfies the Farkas rule.*

Proof. Let $x_0 \in \mathrm{ri}(\mathrm{dom} f \cap C)$ be a weak Slater point of the convex system $\{f_i : i \in I\}$ and let $\hat{I} := \{i_0\} \cup I(x_0)$, where $i_0 \notin I$ is an index, and $I(x_0)$ is the active index set defined by (3.4). Consider the finite system $\{f_i : i \in \hat{I}\}$ with $f_{i_0} := F_0$, where F_0 is the sup-function defined by (3.6). Then $x_0 \in \mathrm{ri}(\mathrm{dom} f \cap C)$ is also a weak Slater point of the finite system $\{f_i : i \in \hat{I}\}$. Clearly, $A = \{x \in C : f_i(x) \le 0, \forall i \in \hat{I}\}$. Given $p \in \mathbb{R}^n$, consider the following convex optimization problem:

$$P_{(f-p)} \qquad \begin{array}{ll} \text{Minimize} & f(x) - \langle p, x\rangle, \\ \text{subject to} & f_i(x) \le 0,\ i \in \hat{I}, \\ & x \in C, \end{array}$$

and its corresponding Lagrangian dual problem:

$$D_{(f-p)} \qquad \begin{array}{ll} \text{Maximize} & \inf\limits_{x\in C} \left(f(x) - \langle p, x\rangle + \sum_{i\in\hat{I}} \lambda_i f_i(x)\right), \\ \text{subject to} & \lambda \in \mathbb{R}^{\hat{I}}_+. \end{array}$$

Denote $\tilde{C} := C \cap \mathrm{dom} f$. Then problems $P_{(f-p)}$ and $D_{(f-p)}$ are equivalent to

$$\begin{array}{ll} \text{Minimize} & f(x) - \langle p, x\rangle, \\ \text{subject to} & f_i(x) \le 0,\ i \in \hat{I}, \\ & x \in \tilde{C}, \end{array} \tag{4.12}$$

and

$$\begin{array}{ll} \text{Maximize} & \inf\limits_{x\in \tilde{C}} \left(f(x) - \langle p, x\rangle + \sum_{i\in\hat{I}} \lambda_i f_i(x)\right), \\ \text{subject to} & \lambda \in \mathbb{R}^{\hat{I}}_+, \end{array} \tag{4.13}$$

respectively. Applying Proposition 4.8 (to $\{\tilde{C}, f_i : i \in \hat{I}\}$ in place of $\{C, f_i : i \in I\}$), we conclude that the strong Lagrangian duality between (4.12) and (4.13) holds for each $p \in \mathbb{R}^n$. Hence, the strong Lagrangian duality between $P_{(f-p)}$ and $D_{(f-p)}$ holds for each $p \in \mathbb{R}^n$. This, together with [12, Theorem 5.2], implies that

$$\mathrm{epi}(f + \delta_A)^* = \bigcup_{\lambda\in\mathbb{R}^{\hat{I}}_+} \mathrm{epi} \left(f + \delta_C + \sum_{i\in\hat{I}} \lambda_i f_i \right)^* . \tag{4.14}$$

Let $S_0 := \{x \in \mathbb{R}^n : f_i(x) \leq 0, \forall\ i \in I \setminus I(x_0)\}$. Then by assumption Remark 3.2 is applicable and so we have that

$$\text{epi}\delta^*_{S_0} \subseteq \text{epi}\delta^*_C + \sum_{i\in I} \text{cone}(\text{epi}f_i^*), \tag{4.15}$$

Since $f_{i_0} = F_0 \leq \delta_{S_0}$, it follows from (2.1) that $\text{cone}(\text{epi}f^*_{i_0}) \subseteq \text{epi}\delta^*_{S_0}$. Hence

$$\text{cone}(\text{epi}f^*_{i_0}) \subseteq \text{epi}\delta^*_{S_0} \subseteq \text{epi}\delta^*_C + \sum_{i\in I} \text{cone}(\text{epi}f_i^*). \tag{4.16}$$

Furthermore, for each $\lambda \in \mathbb{R}^{\hat{I}}_+$, one uses (2.4) and (4.16) to get that

$$\begin{aligned} \text{epi}\left(f + \delta_C + \textstyle\sum_{i\in\hat{I}} \lambda_i f_i\right)^* &= \lambda_{i_0}\text{epi}f^*_{i_0} + \text{epi}\left(f + \delta_C + \textstyle\sum_{i\in I(x_0)} \lambda_i f_i\right)^* \\ &\subseteq \text{epi}\delta^*_C + \textstyle\sum_{i\in I} \text{cone}(\text{epi}f_i^*) + \text{epi}\left(f + \delta_C + \textstyle\sum_{i\in I(x_0)} \lambda_i f_i\right)^* \\ &\subseteq \cup_{\lambda\in\mathbb{R}^{(I)}_+} \text{epi}\left(f + \delta_C + \textstyle\sum_{i\in I} \lambda_i f_i\right)^*. \end{aligned} \tag{4.17}$$

where the last inclusion is because of (2.3). Thus, combining (4.14) and (4.17) gives that

$$\text{epi}(f + \delta_A)^* \subseteq \bigcup_{\lambda\in\mathbb{R}^{(I)}_+} \text{epi}\left(f + \delta_C + \sum_{i\in I} \lambda_i f_i\right)^*.$$

Hence (4.6) holds, and the conclusion follows from Proposition 4.5. The proof is complete. □

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