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ON CONVERGENCE OF A TRUNCATED GAUSS-NEWTON METHOD FOR SOLVING UNDERDETERMINED NONLINEAR LEAST SQUARES PROBLEMS

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Abstract. We consider a truncated Gauss-Newton method for solving nonlinear least squares problems (NLSP) for the underdetermined case. Under some mild conditions, the method converges to a solution at rate of ν when the involved parameter ν in the truncated method satisfies $\nu \in (1,2]$, and superlinearly when $\nu = 1$ and $\theta_k \to 0$. It should be remarked that our techniques for convergence analysis are quite different from that used in [Appl. Numer. Math., 111, 92–110 (2017)].

1. Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous Fréchet differentiable function with its Fréchet derivative denoted by f'. Consider the following nonlinear least squares problem (NLSP)

$$\min_{x\in\mathbb{R}^n} \phi(x) := \frac{1}{2}\|f(x)\|^2, \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean norm. Applications of this kind problem can be found in chemistry, physics, finance, economics and so on; see [3, 8, 15, 19] and references therein.

Newton's method is one of the most important algorithms for solving NLSP (1.1) (see [1, 2, 6, 7, 13, 19, 22] and references therein), which in general converges quadratically. However, it requires the computation of the Hessian matrix of ϕ at each iteration, which may cost expensive, especially for large scale problems. In order to make the procedure more efficient, Gauss-Newton (GN) method (see [11, 16, 18]) is proposed to obtain the search direction d_k by solving the following normal equations:

$$f'(x_k)^T f'(x_k) d = -f'(x_k)^T f(x_k). \tag{1.2}$$

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That is, the GN iterates is given by

$$x_{k+1} := x_k - f'(x_k)^{\dagger} f(x_k), \tag{1.3}$$

where $f'(x_k)^{\dagger}$ is the Moore-Penrose inverse of $f'(x_k)$. The local as well as semi-local convergence properties of the GN method have been explored extensively; see for example [9, 11, 14, 16–18, 20].

In the present paper, we particularly focus on the NLSP (1.1) for the underdetermined case (i.e., $m \le n$), which is found to be applicable in various areas; see [4, 5, 10, 18, 20] and references therein. Note that the underlying problem size of (1.2) will be large in the case when $m \ll n$. In order to overcome this disadvantage, under the full row rank assumption of the Jacobian $f'(x_k)$, Bao et al. [4] proposed another approach to obtain d_k, in which one first finds s_k by solving the subproblem

$$f'(x_k)f'(x_k)^T s = -f(x_k)$$

and then takes $d_k := f'(x_k)^T s_k$. Based on this technique, we consider the following inexact truncated GN method for solving underdetermined NLSP.

Algorithm 1.1

Step 0: Choose an initial point $x_0 \in \mathbb{R}^n$, $\epsilon > 0$, $\nu \in [1, 2]$, and a non-negative sequence $\{\theta_k\}$. Set $k := 0$.

Step 1: If $\|f(x_k)\| \le \epsilon$, then stop.

Step 2: Approximately solve

$$f'(x_k)f'(x_k)^T s = -f(x_k) \tag{1.4}$$

to find s_k to satisfy

$$\|r_k\| \le \theta_k \|f(x_k)\|^{\nu}, \tag{1.5}$$

where $r_k := f'(x_k)f'(x_k)^T s_k + f(x_k)$.

Step 3: Set $d_k := f'(x_k)^T s_k$ and $x_{k+1} := x_k + d_k$. Set $k := k + 1$ and go to Step 1.

Algorithm 1.1 was proposed in [4] for $\nu = 1$ and 2, but, instead of (1.5), with the following residual controls:

$$\|f'(x_0)^{\dagger} r_k\| \le \theta_k \|f'(x_0)^{\dagger} f(x_k)\| \quad \text{or} \quad \|f'(x_0)^{\dagger} r_k\| \le \theta_k \|f'(x_0)^{\dagger} f(x_k)\|^2 \tag{1.6}$$

for $\nu = 1$ or 2, respectively. Under certain mild conditions, local convergence results of sequences generated by Algorithm 1.1 with (1.6) in place of (1.5) were established in [4]; see [4, Theorems 3.1 and 3.2]. Numerical results presented in [4] showed that Algorithm 1.1 satisfying (1.6), based on the formation (1.4), is more efficient than the methods based on (1.2).

In the present paper, we develop a different technique, which works for all $\nu \in [1, 2]$, to study the local convergence property of Algorithm 1.1. More precisely, under the assumption that f' is Lipschitz continuous around a solution x^* of (1.1) and $f'(x^*)$ is of full row rank, we show that Algorithm 1.1 converges locally to a solution of (1.1). Furthermore, the convergence rate is at least ν if $\nu \in (1, 2]$ and superlinearly if $\nu = 1$ and $\theta_k \to 0$. This extends/improves the corresponding results in [4, Theorems 3.2 and 3.4] where it was only showed that this algorithm converges locally to a solution of (1.1), and the convergence rate is at least quadratic in the case when $\nu = 2$.

The paper is organized as follows. In section 2, we present some notions and preliminary results. In section 3, we use another approach which is quite different from [4] to establish the local convergence rate of Algorithm 1.1 under more general assumption of parameter $\nu \in [1, 2]$.

2. Preliminaries and auxiliary results

Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, we use $\mathbf{B}(x, r)$ (resp. $\overline{\mathbf{B}(x, r)}$) to denote the open (resp. closed) ball with radius r and center x. For $W \subseteq \mathbb{R}^n$, the distance function associated with W and the projection onto W are denoted by $\mathrm{d}(x, W)$ and $P_W(x)$, respectively, and defined by

$$\mathrm{d}(x, W) := \inf\{\|x - y\| \mid y \in W\} \quad \text{and} \quad P_W(x) := \{y \in W \mid \|x - y\| = \mathrm{d}(x, W)\}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous Fréchet differentiable function with its Fréchet derivative denoted by f'. Recall that f' is Lipschitz continuous on $\mathbf{B}(\hat{x}, r)$ with modulus L if

$$\|f'(y) - f'(x)\| \le L\|y - x\| \qquad \text{for each } x, y \in \mathbf{B}(\hat{x}, r), \tag{2.1}$$

and f' is local Lipschitz continuous around $\hat{x}$ if there exist $r, L > 0$ such that f' is Lipschitz continuous on $\mathbf{B}(\hat{x}, r)$ with modulus L.

Remark 2.1. Suppose that f' is Lipschitz continuous on $\mathbf{B}(\hat{x}, r)$ with modulus L. Then, by (2.1), one has that

$$\|f(y) - f(x) - f'(x)(y - x)\| \le \frac{L}{2}\|y - x\|^2 \qquad \text{for each } x, y \in \mathbf{B}(\hat{x}, r) \tag{2.2}$$

and

$$\|f(y) - f(x)\| \le K_r\|y - x\| \qquad \text{for each } x, y \in \mathbf{B}(\hat{x}, r),$$

where $K_r := \sup_{x \in \mathbf{B}(\hat{x}, r)} \|f'(x)\|$.

Let $A \in \mathbb{R}^{m\times n}$ be an $m \times n$ real matrix and $A^T \in \mathbb{R}^{n\times m}$ denote its transpose. We say that $A^\dagger \in \mathbb{R}^{n\times m}$ is the Moore-Penrose inverse of A if it satisfies the following four equalities:

$$AA^\dagger A = A, \; A^\dagger AA^\dagger = A^\dagger, \; (AA^\dagger)^T = AA^\dagger, \; (A^\dagger A)^T = A^\dagger A.$$

In particular, if A has full row rank, then

$$A^\dagger = A^T(AA^T)^{-1} \quad \text{and} \quad AA^\dagger = I_m, \tag{2.3}$$

where $I_m \in \mathbb{R}^{m\times m}$ is the identity matrix. Moreover, by the definition of the Moore-Penrose inverse, one can easily check that $(A^\dagger B)^\dagger = B^\dagger A$ if both A and B have full row rank; see [12, 21] for more details.

The following lemma about the matrix perturbation is well known (see [12, 21]).

Lemma 2.2. *Let $A, B \in \mathbb{R}^{m\times n}$ be matrices. Assume that*

$$1 \le \operatorname{rank}(A) \le \operatorname{rank}(B) \quad \textit{and} \quad \|B^\dagger\| \cdot \|A - B\| < 1.$$

Then

$$\operatorname{rank}(A)=\operatorname{rank}(B) \quad \textit{and} \quad \|A^\dagger\| \le \frac{\|B^\dagger\|}{1-\|B^\dagger\|\cdot\|A-B\|}.$$

Throughout the whole paper, the solution set of the equation $f(x)=0$ is denoted by S, namely

$$S := \{x|\ f(x)=0\}.$$

Fixing the triple $(x^*;\bar r, L)$ with $x^*\in\mathbb{R}^n$ and $\bar r, L\in(0,+\infty)$, we consider the following assumption for f associated with the triple $(x^*;\bar r, L)$:
(2.4)

- $f'(x^*)$ is of full row rank;
- $f'(\cdot)$ is Lipschitz continuous on $\mathbf{B}(x^*,\bar r)$ with modulus L.

Below, we recall the following proposition on the convergence property of GN method (1.3), which is taken from [16, Corollary 5.1]. Let $\alpha>0$, $\mu>0$ and $t_0^*>0$, and assume that

$$\alpha\mu\le\frac{1}{2} \quad\text{and}\quad t_0^*=\frac{1-\sqrt{1-2\alpha\mu}}{\mu}. \tag{2.5}$$

Obviously,

$$t_0^*=\frac{2\alpha}{1+\sqrt{1-2\alpha\mu}}\le 2\alpha.$$

Proposition 2.3. *Assume* (2.5)*, and let $x_0\in\mathbb{R}^n$ be such that $\|f'(x_0)^\dagger f(x_0)\|\le\alpha$ and $f'(x_0)$ is of full row rank. Suppose that $f'(x_0)^\dagger f'(\cdot)$ is Lipschitz continuous on $\mathbf{B}(x_0,t_0^*)$ with modulus μ. Then, the sequence $\{x_k\}$ generated by GN method* (1.3) *with initial point x_0, converges to a solution $z^*\in S$ and $\|x_0-z^*\|\le t_0^*$; hence* $\mathrm{d}(x_0,S)\le 2\alpha$.

The following proposition is crucial in the convergence analysis in section 3. As usual, we use $\kappa(A):=\|A\|\|A^\dagger\|$ to denote the generalized condition number of a matrix A.

Proposition 2.4. *Let $x^*\in S$. Suppose that f satisfies assumption* (2.4) *associated with $(x^*;\bar r,L)$. Let $r_0=\min\left\{\bar r,\frac{1}{2L\|f'(x^*)^\dagger\|}\right\}$. Then, there exists $\tilde r\in(0,r_0)$ such that, for each $x\in\mathbf{B}(x^*,\tilde r)$, $f'(x)$ is of full row rank, and the following inequalities hold:*

$$\frac{1}{2}\|f'(x)^\dagger\|\le\|f'(x^*)^\dagger\|\le 2\|f'(x)^\dagger\|, \tag{2.6}$$

$$\mathrm{d}(x,S)\le 4\|f'(x^*)^\dagger\|\|f(x)\|, \tag{2.7}$$

and

$$\|f'(y)\|\|f'(x^*)^\dagger\|\le 2\kappa(f'(x))+4 \quad \textit{for each } y\in\mathbf{B}(x^*,r_0). \tag{2.8}$$

Proof. Recall that $K_{r_0}=\sup_{x\in\mathbf{B}(x^*,r_0)}\|f'(x)\|$, and set

$$\tilde r:=\min\left\{\frac{r_0}{2},\frac{1}{4L\|f'(x^*)^\dagger\|},\frac{1}{8LK_{r_0}\|f'(x^*)^\dagger\|^2},\frac{\bar r}{8K_{r_0}\|f'(x^*)^\dagger\|}\right\}. \tag{2.9}$$

Below we will show that $\tilde{r}$ is as desired. To do this, let $x \in \mathbf{B}(x^*, \tilde{r})$. Then,

$$L\|f'(x^*)^\dagger\|\|x - x^*\| < \frac{1}{4}. \tag{2.10}$$

Combining this with assumption (2.4) yields that

$$\|f'(x^*)^\dagger\| \cdot \|f'(x) - f'(x^*)\| \le L\|f'(x^*)^\dagger\|\|x - x^*\| < \frac{1}{2}.$$

Thus, applying Lemma 2.2 to $f'(x)$ and $f'(x^*)$ in place of A and B, respectively, one has that $f'(x)$ is of the same rank as that of $f'(x^*)$ (and so is of full row rank), and

$$\|f'(x)^\dagger\| \le \frac{\|f'(x^*)^\dagger\|}{1 - \|f'(x^*)^\dagger\|\|f'(x) - f'(x^*)\|} \le 2\|f'(x^*)^\dagger\|; \tag{2.11}$$

hence, the first inequality of (2.6) is checked. Note again by assumption (2.4) and (2.11) that

$$\|f'(x)^\dagger\| \cdot \|f'(x^*) - f'(x)\| \le L\|f'(x)^\dagger\|\|x^* - x\| \le 2L\|f'(x^*)^\dagger\|\|x^* - x\| < \frac{1}{2}$$

(due to (2.10)). Thus, applying again Lemma 2.2, we have that

$$\|f'(x^*)^\dagger\| \le \frac{\|f'(x)^\dagger\|}{1 - \|f'(x)^\dagger\| \cdot \|f'(x^*) - f'(x)\|} \le 2\|f'(x)^\dagger\|. \tag{2.12}$$

Then, the second inequality of (2.6) is seen to hold.

To check (2.7), set $\mu := 2L\|f'(x^*)^\dagger\|$ and $\alpha := \|f'(x)^\dagger f(x)\|$. Note by (2.11) and the definition of $K_{\bar{r}}$ that

$$\alpha \le \|f'(x)^\dagger\|\|f(x) - f(x^*)\| \le 2K_{r_0}\|f'(x^*)^\dagger\|\|x - x^*\|, \tag{2.13}$$

and so

$$\alpha\mu \le 4LK_{r_0}\|f'(x^*)^\dagger\|^2\|x - x^*\| \le \frac{1}{2},$$

where the last inequality holds because of (2.9). Below, we show that

$$f'(x)^\dagger f'(\cdot) \text{ is Lipschitz continuous on } \mathbf{B}(x, t_0^*) \text{ with modulus } \mu. \tag{2.14}$$

Granting this, applying Proposition 2.3 to x in place x_0, one has that $\mathrm{d}(x, S) \le 2\alpha$. Note by (2.11) that

$$\alpha = \|f'(x)^\dagger f(x)\| \le 2\|f'(x^*)^\dagger\|\|f(x)\|.$$

Thus, (2.7) is seen to hold. To show (2.14), let $y_1, y_2 \in \mathbf{B}(x, t_0^*)$. Then, for each $i = 1, 2$, it follows from (2.9) that $\|x - x^*\| < \frac{\bar{r}}{2}$ and so

$$\|y_i - x^*\| \le \|y_i - x\| + \|x - x^*\| < t_0^* + \frac{\bar{r}}{2} \le \bar{r},$$

where the last inequality holds because

$$t_0^* \le 2\alpha \le 4K_{r_0}\|f'(x^*)^\dagger\|\|x - x^*\| \le \frac{\bar{r}}{2}$$

(due to (2.13) and (2.9)). Hence, it follows from assumption (2.4) and (2.11) that

$$\|f'(x)^\dagger f'(y_1) - f'(x)^\dagger f'(y_2)\| \le \|f'(x)^\dagger\|\|f'(y_1) - f'(y_2)\| \le 2L\|f'(x^*)^\dagger\|\|y_1 - y_2\|,$$

and so (2.14) is checked by the definition of μ.

Finally, we show that (2.8) holds. Let $y \in \mathbf{B}(x^*, r_0)$. By assumption (2.4), we have that

$$\|f'(y)\| \le \|f'(x)\| + \|f'(y) - f'(x)\| \le \|f'(x)\| + L\|y - x\|. \tag{2.15}$$

Note further that

$$L\|y - x\| \le L(\|y - x^*\| + \|x - x^*\|) \le 2Lr_0 < \frac{1}{\|f'(x^*)^\dagger\|} \le \frac{2}{\|f'(x)^\dagger\|},$$

where the third inequality holds by (2.9) and the last by (2.11). Combining this with (2.15) yields that

$$\|f'(y)\| \le \|f'(x)\| + \frac{2}{\|f'(x)^\dagger\|}.$$

This, together with (2.12), implies that

$$\|f'(y)\|\|f'(x^*)^\dagger\| \le \left(\|f'(x)\| + \frac{2}{\|f'(x)^\dagger\|}\right) 2\|f'(x)^\dagger\| = 2\kappa(f'(x)) + 4.$$

Thus, (2.8) is seen to hold and the proof is complete. □

3. Local convergence analysis of Algorithm 1.1

In this section, we show that a sequence generated by Algorithm 1.1 converges to a solution at rate of ν when $\nu \in (1, 2]$, and superlinearly when $\nu = 1$ and $\theta_k \to 0$. Let $\{x_k\}$ be a sequence generated by Algorithm 1.1 with initial piont x_0 (together with the associated sequence $\{d_k\}$). In view of Algorithm 1.1, one has that, for each $k \in \mathbb{N}$,

$$x_{k+1} = x_k + d_k \quad \text{and} \quad d_k = f'(x_k)^T s_k = f'(x_k)^\dagger(-f(x_k) + r_k). \tag{3.1}$$

Throughout the whole paper, we always assume that

$$\theta := \sup_{k \ge 0} \theta_k < +\infty, \tag{3.2}$$

and recall that $S = \{x |\ f(x) = 0\}$. The following lemma is about some properties related to the sequences $\{x_k\}$ and $\{d_k\}$.

Lemma 3.1. *Assume* (3.2) *and let* $x^* \in S$. *Suppose that* f *satisfies assumption* (2.4) *associated with* $(x^*; \bar{r}, L)$. *Let* $r_0 = \min\left\{\bar{r}, \frac{1}{2L\|f'(x^*)^\dagger\|}\right\}$ *and* $K_{r_0} = \sup_{x\in\mathbf{B}(x^*,r_0)} \|f'(x)\|$. *Then, there exist positive constant* $c > 0$ *and* $0 < r_1 \le r_0$ *such that the following two assertions hold:*

(i) *If* $x_k \in \mathbf{B}(x^*, r_1)$, *then*

$$\|d_k\| \le c\mathrm{d}(x_k, S). \tag{3.3}$$

(ii) *If* $x_k, x_{k+1} \in \mathbf{B}(x^*, r_1)$, *then*

$$\mathrm{d}(x_{k+1}, S) \le \begin{cases} c\mathrm{d}(x_k, S)^\nu, & \nu \in (1, 2]; \\ c(\mathrm{d}(x_k, S) + \theta_k)\mathrm{d}(x_k, S), & \nu = 1; \end{cases} \tag{3.4}$$

hence,

$$\mathrm{d}(x_{k+1}, S) \le \frac{1}{2}\mathrm{d}(x_k, S) \tag{3.5}$$

if it is additionally assumed that $\theta_k \le \frac{1}{16K_{r_0}\|f'(x^*)^\dagger\|}$ *for* $\nu = 1$.

Proof. Write

$$c_1 := 2\|f'(x^*)^\dagger\|\left(K_{r_0} + \theta K_{r_0}^\nu\right) \quad \text{and} \quad c_2 := 2\|f'(x^*)^\dagger\|\left(Lc_1^2 + 2\theta K_{r_0}^\nu\right).$$

Take

$$c := \max\left\{c_1, c_2, 2Lc_1^2\|f'(x^*)^\dagger\|, 4K_{r_0}\|f'(x^*)^\dagger\|\right\}. \tag{3.6}$$

Note that Proposition 2.4 is applicable to concluding that there exists $\tilde{r} > 0$ such that all the conclusions of Proposition 2.4 hold. Set

$$r_1 := \min\left\{\frac{\bar{r}}{4}, \frac{1}{2L\|f'(x^*)^\dagger\|}, \tilde{r}, \frac{1}{4c}, \left(\frac{1}{2c}\right)^{\frac{1}{\nu-1}}\right\}. \tag{3.7}$$

By the definition of K_{r_0}, we have that $c \ge 4$, and so $0 < r_1 < 1$. Below we show that c and r_1 are as desired. To do this, let $x_k \in \mathbf{B}(x^*, r_1)$ and let $\bar{x}_k \in P_S(x_k)$. Then, we have

$$\mathrm{d}(x_k, S) = \|x_k - \bar{x}_k\| \le \|x_k - x^*\| \le r_1 < 1 \tag{3.8}$$

and

$$\|\bar{x}_k - x^*\| \le \|\bar{x}_k - x_k\| + \|x_k - x^*\| \le 2\|x_k - x^*\| \le 2r_1 < \bar{r}.$$

Then, it follows from the definition of K_{r_0} that

$$\|f(x_k)\| = \|f(x_k) - f(\bar{x}_k)\| \le K_{r_0}\|x_k - \bar{x}_k\| = K_{r_0}\mathrm{d}(x_k, S). \tag{3.9}$$

This, together with (1.5), implies that

$$\|r_k\| \le \theta_k\|f(x_k)\|^\nu \le \theta_k(K_{r_0}\mathrm{d}(x_k, S))^\nu. \tag{3.10}$$

Since $\|x_k - x^*\| < \tilde{r}$ (due to (3.7)), it follows from (3.1) and (2.6) that

$$\|d_k\| \le \|f'(x_k)^\dagger\|(\|f(x_k)\| + \|r_k\|) \le 2\|f'(x^*)^\dagger\|(\|f(x_k)\| + \|r_k\|).$$

This, together with (3.10) and (3.9), implies that

$$\begin{aligned} \|d_k\| &\le 2\|f'(x^*)^\dagger\|(K_{r_0}\mathrm{d}(x_k, S) + \theta_k K_{r_0}^\nu \mathrm{d}(x_k, S)^\nu) \\ &\le 2\|f'(x^*)^\dagger\|(K_{r_0} + \theta_k K_{r_0}^\nu)\mathrm{d}(x_k, S) \\ &= c_1 \mathrm{d}(x_k, S), \end{aligned} \tag{3.11}$$

where the second inequality holds because of (3.8) and $\nu \ge 1$, and so (3.3) is checked by the definition of c.

To check (3.4), noting by (3.1) and (2.3), one has that

$$\|f(x_k) + f'(x_k)d_k\| = \|r_k\|.$$

Combing this with (3.10) yields that

$$\|f(x_k) + f'(x_k)d_k\| \le \theta_k K_{r_0}^\nu \mathrm{d}(x_k, S)^\nu. \tag{3.12}$$

Since $x_k, x_{k+1} \in \mathbf{B}(x^*, r_1)$ and $x_{k+1} = x_k + d_k$, it follows from (2.2) and (3.12) that

$$\begin{aligned} \|f(x_{k+1})\| &\le \|f(x_k + d_k) - f(x_k) - f'(x_k)d_k\| + \|f(x_k) + f'(x_k)d_k\| \\ &\le \tfrac{L}{2}\|d_k\|^2 + \theta_k K_{r_0}^\nu \mathrm{d}(x_k, S)^\nu \\ &\le \tfrac{Lc_1^2}{2}\mathrm{d}(x_k, S)^2 + \theta_k K_{r_0}^\nu \mathrm{d}(x_k, S)^\nu, \end{aligned}$$

where the last inequality holds because of (3.11). Combining this with (2.7) yields that

$$\mathrm{d}(x_{k+1}, S) \leq 4\|f'(x^*)^\dagger\|\|f(x_{k+1})\| \leq 4\|f'(x^*)^\dagger\| \left(\frac{Lc_1^2}{2}\mathrm{d}(x_k, S)^2 + \theta_k K_{r_0}^\nu \mathrm{d}(x_k, S)^\nu\right).$$

Thus, for the case when $\nu = 1$, (3.4) follows directly from (3.6), while for the case when $\nu \in (1, 2]$, (3.4) follows from (3.8), thanks to the definitions of θ and c_2, and (3.4) is proved.

Finally, we check (3.5). In the case when $\nu = 1$ and $\theta_k \leq \frac{1}{16K_{r_0}\|f'(x^*)^\dagger\|}$, it follows that $4K_{r_0}\theta_k\|f'(x^*)^\dagger\| \leq \frac{1}{4}$. Noting further that $\mathrm{d}(x_k, S) \leq r_1 \leq \frac{1}{4c}$ (due to (3.8) and (3.7)), one has from the second inequality of (3.4) that

$$\mathrm{d}(x_{k+1}, S) \leq \frac{1}{4}\mathrm{d}(x_k, S) + \frac{1}{4}\mathrm{d}(x_k, S) = \frac{1}{2}\mathrm{d}(x_k, S).$$

In the case when $\nu \in (1, 2]$, it follows from the first inequality of (3.4) that

$$\mathrm{d}(x_{k+1}, S) \leq c\mathrm{d}(x_k, S)^\nu \leq c\mathrm{d}(x_k, S)^{\nu-1}\mathrm{d}(x_k, S) \leq cr_1^{\nu-1}\mathrm{d}(x_k, S) \leq \frac{1}{2}\mathrm{d}(x_k, S),$$

where the last inequality holds because of (3.7). Hence, (3.5) is seen to hold. The proof is completed. □

Lemma 3.2. *Assume* (3.2) *and let $x^* \in S$. Suppose that f satisfies assumption* (2.4) *associated with $(x^*; \bar{r}, L)$. Then, for any $r > 0$, there exist $\hat{r} > 0$ such that, for any $x_0 \in \mathbf{B}(x^*, \hat{r})$, any sequence $\{x_k\}$ generated by Algorithm 1.1 with initial point x_0 stays in $\mathbf{B}(x^*, r)$, and satisfies the following estimate:*

$$\mathrm{d}(x_{k+1}, S) \leq \frac{1}{2}\mathrm{d}(x_k, S) \quad \textit{for each } k \geq 0, \tag{3.13}$$

if it is assumed additionally for $\nu = 1$ that

$$\theta \leq \frac{1}{16(2\kappa(f'(x_0)) + 4)}. \tag{3.14}$$

Proof. Note that Lemma 3.1 and Proposition 2.4 are applicable to concluding that there exist $c, r_1, \tilde{r}$ such that all the conclusions of Lemma 3.1 and Proposition 2.4 hold. Let $r > 0$. Without loss of generality, we assume that $r \leq \min\{r_1, \tilde{r}\}$. Let $\hat{r} := \frac{r}{1+2c}$. Let $x_0 \in \mathbf{B}(x^*, \hat{r})$. Below, we show by mathematical induction that $x_k \in \mathbf{B}(x^*, r)$ for each $k \geq 0$ and (3.13) holds. In fact, by definition of $\hat{r}$, $x_0 \in \mathbf{B}(x^*, r)$ and so (3.3) holds for $k = 0$. This implies that

$$\|x_1 - x^*\| \leq \|x_1 - x_0\| + \|x_0 - x^*\| \leq \|d_0\| + \hat{r} \leq c\mathrm{d}(x_0, S) + \hat{r} \leq (1 + c)\hat{r} \leq r \leq r_1.$$

Note by (2.8) that $\frac{1}{16(2\kappa(f'(x_0))+4)} \leq \frac{1}{16K_{r_0}\|f'(x^*)^\dagger\|}$, where $K_{r_0} = \sup_{x\in\mathbf{B}(x^*,r_0)} \|f'(x)\|$ and $r_0 = \min\left\{\bar{r}, \frac{1}{2L\|f'(x^*)^\dagger\|}\right\}$. Thus, in the case when $\nu = 1$, (3.14) implies that

$$\theta = \sup_{k\geq 0} \theta_k \leq \frac{1}{16K_{r_0}\|f'(x^*)^\dagger\|}. \tag{3.15}$$

Consequently, (3.5) holds for $k = 0$, that is, (3.13) holds for $k = 0$. Assume that $x_0, \cdots, x_k \in \mathbf{B}(x^*, r)$ and (3.13) holds for $0, 1, \cdots, k-1$. Then, (3.3) holds for

$0, 1, \cdots, k$. Consequently, it follows that

$$\begin{aligned}\|x_{k+1} - x^*\| &\le \|x_0 - x^*\| + \sum_{i=0}^{k} \|d_i\| \le \hat{r} + c \sum_{i=0}^{k} \mathrm{d}(x_i, S) \\ &\le \hat{r} + c\hat{r} \sum_{i=0}^{k} \left(\frac{1}{2}\right)^i \le (1 + 2c)\, \hat{r} \le r\end{aligned}$$

(by definition of $\hat{r}$). Hence, $x_{k+1} \in \mathbf{B}(x^*, r)$. This, together with (3.15), implies that (3.5) holds for k, that is, (3.13) holds for k. This completes the proof. $\square$

Now, we are ready to present the following local convergence result of Algorithm 1.1.

Theorem 3.3. *Assume* (3.2) *and let* $x^* \in S$. *Suppose that* f *satisfies assumption* (2.4) *associated with* $(x^*; \bar{r}, L)$. *Then, there exists* $\hat{r} > 0$ *such that, for any* $x_0 \in \mathbf{B}(x^*, \hat{r})$, *any sequence* $\{x_k\}$ *generated by Algorithm 1.1 with initial point* x_0 *converges to some point* $\bar{x} \in S$ *if* (3.14) *is assumed additionally for* $\nu = 1$. *Moreover, one has the following convergence rates:*

(i) *If* $\nu = 1$ *and* $\theta_k \to 0$, *then the convergence rate of* $\{x_k\}$ *is at least superlinear:*

$$\lim_{k \to \infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|} = 0.$$

(ii) *If* $\nu \in (1, 2]$, *then the convergence rate of* $\{x_k\}$ *is at least* ν*:*

$$\limsup_{k \to \infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|^{\nu}} \le +\infty.$$

Consequently, if $\nu = 2$, *then the convergence rate is at least quadratic.*

Proof. Since f' is local Lipschitz continuous around x^*, there exist $\bar{r}, L > 0$ such that f' is Lipschitz continuous on $\mathbf{B}(x^*, \bar{r})$ with modulus L. Hence, f satisfies assumption (2.4) associated with $(x^*; \bar{r}, L)$. Thus, Lemmas 3.1 and 3.2 are applicable to concluding that there exist $c, r_1, \hat{r}$ such that if $x_0 \in \mathbf{B}(x^*, \hat{r})$, then $x_k \in \mathbf{B}(x^*, r_1)$ for each $k \ge 0$, and (3.3), (3.4) and (3.13) hold for each $k \ge 0$. Hence, it follows from (3.3) and (3.13) that

$$\sum_{k=0}^{\infty} \|d_k\| \le c \sum_{k=0}^{\infty} \mathrm{d}(x_k, S) \le c\hat{r} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \le 2c\hat{r} < +\infty. \tag{3.16}$$

This means that $\{x_k\}$ is a Cauchy sequence. Suppose that $\{x_k\}$ converges to some point $\bar{x}$. Note further by (3.16) that

$$\lim_{k \to \infty} \mathrm{d}(x_k, S) = 0. \tag{3.17}$$

As S is closed, it follows that $\bar{x} \in S$. Below, we divide the proof into two cases.

Case 1. $\nu \in (1, 2]$. We show that there exists a positive integer N_1 such that for all $k \ge N_1$,

$$\|d_{k+1}\| \le 2^{\nu} c^2 \|d_k\|^{\nu} \tag{3.18}$$

and

$$\lim_{k\to\infty} \frac{\|\sum_{i=k+1}^{\infty} d_i\|}{\|d_{k+1}\|} = 1. \tag{3.19}$$

Granting this, we have that

$$\lim_{k\to\infty} \frac{\|x_{k+1}-\bar{x}\|}{\|x_k-\bar{x}\|^{\nu}} = \lim_{k\to\infty} \frac{\|\sum_{i=k+1}^{\infty} d_i\|}{\|\sum_{i=k}^{\infty} d_i\|^{\nu}} = \lim_{k\to\infty} \frac{\|d_{k+1}\|}{\|d_k\|^{\nu}} \le 2^{\nu}c^2,$$

which implies that $\{x_k\}$ converges to $\bar{x}$ at rate of ν. To proceed, note that

$$\mathrm{d}(x_k,S) \le \mathrm{d}(x_{k+1},S) + \|x_{k+1}-x_k\| = \mathrm{d}(x_{k+1},S) + \|d_k\|. \tag{3.20}$$

Combining this with (3.13) yields that $\mathrm{d}(x_k,S) \le 2\|d_k\|$. This, together with (3.3) and (3.4), gives that

$$\|d_{k+1}\| \le c\mathrm{d}(x_{k+1},S) \le c^2\mathrm{d}(x_k,S)^{\nu} \le 2^{\nu}c^2\|d_k\|^{\nu},$$

which means that (3.18) holds. Note that (3.16) implies $\lim_{k\to\infty}\|d_k\| = 0$, and so there exists a positive integer N_1, such that for each $k \ge N_1$, $p := 2^{\nu}c^2\|d_k\|^{\frac{\nu-1}{3}} < 1$. This, together with (3.18), yields that

$$\|d_{k+1}\| \le p\|d_k\|^{1+\frac{2\nu-2}{3}} \quad \text{for each } k \ge N_1. \tag{3.21}$$

Fix $k \ge N_1$. It follows inductively from (3.21) that for each $i \ge 2$,

$$\|d_{k+i}\| \le p^{\frac{\left(1+\frac{2\nu-2}{3}\right)^{i-1}-1}{\frac{2\nu-2}{3}}} \|d_{k+1}\|^{\left(1+\frac{2\nu-2}{3}\right)^{i-1}}.$$

This, together with (3.16), implies that

$$\lim_{k\to\infty} \sum_{i=2}^{\infty} \frac{\|d_{k+i}\|}{\|d_{k+1}\|} \le \lim_{k\to\infty} \sum_{i=2}^{\infty} \left(p^{\frac{3}{2\nu-2}}\|d_{k+1}\|\right)^{\left(1+\frac{2\nu-2}{3}\right)^{i-1}-1} = 0. \tag{3.22}$$

Observe further that

$$1 - \frac{\sum_{i=k+2}^{\infty}\|d_i\|}{\|d_{k+1}\|} \le \frac{\|\sum_{i=k+1}^{\infty} d_i\|}{\|d_{k+1}\|} \le 1 + \frac{\sum_{i=k+2}^{\infty}\|d_i\|}{\|d_{k+1}\|}. \tag{3.23}$$

Hence, (3.19) follows directly from (3.22) and (3.23). This completes the proof of (ii).

Case 2. $\nu = 1$ and $\theta_k \to 0$. Note by (3.20) and (3.13) that $\mathrm{d}(x_k,S) \le 2\|d_k\|$ for each $k \ge 0$. Combining this with (3.3) and (3.4) yields that, for each $k \ge 0$,

$$\|d_{k+1}\| \le c\mathrm{d}(x_{k+1},S) \le 2c^2\left(\mathrm{d}(x_k,S)+\theta_k\right)\|d_k\|. \tag{3.24}$$

Let $\epsilon \in (0,1)$. By (3.17) and the fact that $\theta_k \to 0$, there exists a positive integer K, such that for each $k \ge K$, $2c^2\left(\mathrm{d}(x_k,S)+\theta_k\right) \le \epsilon$ and so it follows from (3.24) that $\|d_{k+1}\| \le \epsilon\|d_k\|$ for each $k \ge K$. This implies that, for each $k \ge K$,

$$0 \le \frac{\sum_{i=k+2}^{\infty}\|d_i\|}{\|d_{k+1}\|} \le \frac{\|d_{k+1}\|\sum_{i=1}^{\infty}\epsilon^i}{\|d_{k+1}\|} \le \frac{\epsilon}{1-\epsilon}. \tag{3.25}$$

As $0 < \epsilon < 1$ is arbitrary, letting $\epsilon \to 0$ and $k \to +\infty$ in (3.25), we obtain that

$$\lim_{k\to\infty} \frac{\sum_{i=k+2}^{\infty}\|d_i\|}{\|d_{k+1}\|} = 0.$$

This, together with (3.23), yields that (3.19) holds. Thus, we have

$$\lim_{k\to\infty} \frac{\|x_{k+1}-\bar{x}\|}{\|x_k-\bar{x}\|} = \lim_{k\to\infty} \frac{\|\sum_{i=k+1}^{\infty} d_i\|}{\|\sum_{i=k}^{\infty} d_i\|} = \lim_{k\to\infty} \frac{\|d_{k+1}\|}{\|d_k\|} = 0,$$

which implies that $\{x_k\}$ converges to $\bar{x}$ superlinearly. This completes the proof. □

Remark 3.4. Note that in the case when $\nu = 2$, that is, the residuals control (1.5) is reduced to the following one:

$$\|r_k\| \le \theta_k \|f(x_k)\|^2 \quad \text{for each } k \ge 0.$$

As pointed out in [4, p. 108] that in the case when $f'(x_0)$ is of full row rank, then the residual $\|f'(x_0)^\dagger r_k\| \le \theta_k \|f'(x_0)^\dagger f(x_k)\|^2$ is equivalent to $\|r_k\| \le \eta_k \|f(x_k)\|^2$ (with possible different constants $\{\eta_k\}$). Thus, the local convergence result of [4, Theorem 3.4] follows from Theorem 3.3.

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