WEAK SHARP MINIMA FOR CONVEX INFINITE OPTIMIZATION PROBLEMS IN NORMED LINEAR SPACES

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Abstract. We study in the present paper the characterization issue of the weak sharp minima properties for convex infinite optimization problems in normed linear spaces. We developed a new approach to establish several complete geometric characterizations for the global/bounded/local weak sharp minima property, which extend/improve the corresponding ones in this direction by removing/relaxing the key topological assumptions made on the index set. As by-products, some complete characterizations of the global/bounded/local weak sharp minima are obtained for a subset of the level set of a given convex function (not necessarily the level set itself) in terms of the normal cones and the subdifferentials of the involved convex subset and convex function. These characterization results are of independent interest in extending/improving the existing ones on characterizing the weak sharp minima for convex optimization problems.

Key words. weak sharp minima, convex optimization, infinite optimization problems, ε-subdifferentials

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1. Introduction.

Let X be a normed space, Ω be a nonempty closed and convex subset of X, and Y be an arbitrary index set. Let $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous (in short, lsc) function, and let $\phi_{(\cdot)}(\cdot) : X \times Y \to \overline{\mathbb{R}}$ be such that the function $x \mapsto \phi_y(x)$ is lsc for each $y \in Y$. In the present paper, we consider the following infinite optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \Omega, \\ & \phi_y(x) \le 0, \quad \forall y \in Y. \end{array} \tag{1.1}$$

This kind of problems arises in many practical applications, such as engineering design [31, 32], control of robots [17, 34], data envelopment analysis [24], statistics [12] and social sciences [19, 35] (see also the survey paper [20] and books [1, 4, 15, 16, 33]), and has become an active research area in mathematical programming; see [25, 26, 27, 28, 29, 30, 36, 40, 41, 42, 45]

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and references therein. Our interest in the present paper is focused on the convex infinite optimization problem, that is, we assume, throughout the whole paper, for problem (1.1) that

- Ω is closed and convex;
- $f : X \to \overline{\mathbb{R}}$ and $\phi_y : X \to \overline{\mathbb{R}}$ are proper, lsc and convex for each $y \in Y$.

Let S denote the *solution set* of problem (1.1), that is,

$$S := \{z \in Z | f(z) = \inf_{x \in Z} f(x)\}, \tag{1.2}$$

where $Z := \{x \in \Omega | \phi_y(x) \leq 0 \text{ for each } y \in Y\}$. Let $\alpha > 0$ and $x_0 \in S$. Then, following [42], problem (1.1) is said to have the (global) weak sharp minima property with modulus $\alpha > 0$ if

$$\alpha \mathrm{d}_S(x) \leq f(x) - f(x_0) + \sup_{y \in Y} [\phi_y]_+(x) + \mathrm{d}_\Omega(x) \quad \text{for each } x \in X.$$

One important issue in the development of infinite optimization problem is to provide the geometric characterizations for the weak sharp minima property, especially in terms of the normal cones and the subdifferentials of the involved convex subsets and convex functions. Under the assumptions that

(A1) Y is a compact metric space such that $\phi_{(\cdot)}(x)$ is continuous for each $x \in X$, and
(A2) $\phi_{(\cdot)}(\cdot)$ is real-valued and jointly upper semi-continuous (in short, usc) on $X \times Y$,

Zheng and Yang established in [42, Theorem 3.4] the following characterization result for the weak sharp minima property for problem (1.1) in a Banach space.

Theorem 1.1. *Problem* (1.1) *has the weak sharp minima property with modulus* α *if and only if*

$$\alpha \mathbb{B}^* \cap \mathrm{N}_S(z) \subseteq \partial f(z) + [0,1] \mathrm{cl}^* \mathrm{co}(\cup_{y \in Y_0(z)} \partial \phi_y(z)) + \mathrm{N}_\Omega(z) \cap \mathbb{B}^* \quad \textit{for each } z \in \mathrm{bd} S,$$

where $Y_0(z) := \{y \in Y | \phi_y(z) = 0\}$ *and* $\mathrm{bd}S$ *denotes the boundary of* S.

The study of the weak sharp minima property, as well as the boundedly and local weak sharp minima properties, for the infinite optimization problem seems limited, and, to the best of our knowledge, it has not been founded for the case when the topological assumptions on the index set Y in (A1) and/or (A2) are dropped. This motivates us to investigate the geometric characterization problem of the weak sharp minima property for convex infinite optimization problem (1.1) without any topological assumptions made on Y. Using the ε-subdifferentials, instead of the subdifferentials, of the involved functions, we establish some characterization results similar to Theorem 1.1 for the weak sharp minima property (and also for the boundedly or local weak sharp minima property) for the general problem (1.1); see Theorems 4.8-4.10. In particular, as a consequence, we show that Theorem 1.1 remains true in normed linear spaces under the following weaker assumptions:

(B1) Y is a separated compact topological space, and $\phi_{(\cdot)}(x)$ is usc on Y for each $x \in X$;
(B2) $\phi_y(\cdot)$ is continuous on S for each $y \in Y$.

(Noting that the real-valuedness assumption for $\phi_{(\cdot)}(\cdot)$ in (A2) implies (B2)). The results obtained in the present paper contain not only the complete characterizations for the (global) weak sharp minima property, but also the ones for the bounded and/or local weak sharp minima property, most of which seem new.

To furniture the establishment of the proposed results, we shall first introduce and investigate the notion of weak sharp minima for a subset S_0 of the level set $\mathrm{L}_f(\lambda) := \{x \in X | f(x) = \lambda\}$ (not necessarily the level set $\mathrm{L}_f(\lambda)$ itself): S_0 is said to be a set of the (global) weak sharp minima for the function f with modulus $\alpha > 0$ if

$$f(z) \geq \lambda + \alpha \mathrm{d}_{S_0}(z) \quad \text{for each } z \in X.$$

The notions of boundedly weak sharp minima and local weak sharp minima for the function f are defined similarly; see Definition 3.1 for details. In the special case when $S_0 = \bar{S} := \mathrm{argmin}_{x\in X} f(x)$, the notion of weak sharp minima for f is reduced to the classical one for the following optimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in X. \end{array} \tag{1.3}$$

The notion of the classical (global) weak sharp minima for optimization problem (1.3) introduced by Ferris in [13], as well as the notions of boundedly weak sharp minima and local weak sharp minima introduced by Burke and Deng in [5], have been extensively studied and widely applied in the sensitivity and convergence analysis of many optimization algorithms; see [2, 6, 7, 8, 9, 14, 21, 22, 23, 37, 38, 41, 44] and references therein. In particular, the theory regarding the characterization of the (global) weak sharp minima for problem (1.3) has been well established; see [5, 8, 39] and references therein. In fact, as presented in [39, Theorem 3.10.1] and [5, Theorem 2.3], one has the following complete characterizations for the solution set to be a set of weak sharp minima, where $\mathrm{J}(\cdot)$ is the duality mapping, $\mathrm{P}(\cdot|\bar{S})$ denotes the projection onto $\bar{S}$ and $\mathrm{T}_{\bar{S}}(\cdot)$ denotes the tangent cone of $\bar{S}$; see section 2 for details.

THEOREM 1.2. *Let $\alpha > 0$ and $\bar{S} := \mathrm{argmin}_{x\in X} f(x)$. Consider the following statements:*
(i) *$\bar{S}$ is a set of weak sharp minima for f with modulus α.*
(ii) *For any $x \in X$ and $p \in \mathrm{P}(x|\bar{S})$, $f'(p; x - p) \geq \alpha \mathrm{d}_{\bar{S}}(x)$.*
(iii) *For any $\bar{x} \in \bar{S}$ and $u \in \mathrm{J}^{-1}(\mathrm{N}_{\bar{S}}(\bar{x}))$, $f'(\bar{x}; u) \geq \|u\|$.*
(iv) *For any $\bar{x} \in \bar{S}$ and $u \in \mathrm{J}^{-1}(\mathrm{N}_{\bar{S}}(\bar{x})) \cap \mathrm{cone}(\mathrm{dom} f - \bar{x})$, $f'(\bar{x}; u) \geq \|u\|$.*
(v) *For any $\bar{x} \in \bar{S}$ and $u \in X$, $f'(\bar{x}; u) \geq \alpha \mathrm{d}_{\mathrm{T}_{\bar{S}}(\bar{x})}(u)$.*
(vi) *For any $\bar{x} \in \bar{S}$, $\alpha \mathbb{B}^* \cap \mathrm{N}_{\bar{S}}(\bar{x}) \subseteq \partial f(\bar{x})$.*
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) *and* (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). *If X is a reflexive Banach space, then all above statements are equivalent.*

However, to the best of our knowledge, the theory of characterizations for either bounded or local weak sharp minima for problem (1.3) is still far from being complete as that for the global one, even in the reflexive Banach space setting; see the explanations before Theorem 3.8. As by-products, we will also establish the complete characterization results, similar to the ones as presented in Theorem 1.2, for a subset $S_0 \subseteq \mathrm{L}_f(\lambda)$ to be a set of global/boundedly/local weak sharp minima for f; see Theorems 3.6-3.9. Our characterization results are of independent interest in the development of weak sharp minima for convex optimization problem (1.3) and play a key role in the sequel study of the weak sharp minima property for the infinite convex optimization problem (1.1). In particular, these results not only extend Theorem 1.2 for the weak sharp minima from the special case when $S_0 = \bar{S}$ to the general case when S_0 is a subset of a level set, but also provide complete characterizations for the boundedly and/or local weak sharp minima, which particularly improve the existing corresponding results even in the special case when $S_0 = \bar{S}$; see the explanations before Theorems 3.6 and 3.8 for details. The approaches used here for treating the weak sharp minima property for f or problem (1.1) deviate from that used in [5, 8, 39]: for the function f, we establish the relationships between the equivalent characterizations of weak sharp minima for a subset S_0 of the level set and the ones for the minima set $\bar{S}$ (see Proposition 3.4); for problem (1.1), we introduce an auxiliary function to develop a bridge connecting the weak sharp minima property for problem (1.1) and the weak sharp minima for the auxiliary function over the set S defined as in (1.2) (see Proposition 4.2).

The present paper is organized as follows. In section 2, we present some basic notations and preliminary results used in what follows, mostly for convex functions and sets in normed spaces. Section 3 is devoted to the completed theory of geometric characterizations for a subset of a level set to be a set of global/boundedly/local weak sharp minima for a given function. In section 4, we apply the results obtained in section 3 to investigate the characterization problem of weak sharp minima property for infinite convex optimization problem (1.1).

2. Notation and preliminary results. Let X be a normed space with norm $\|\cdot\|$ and its dual X^*. We use $\langle\cdot,\cdot\rangle$ to denote the canonical pairing between X^* and X, that is, $\langle x^*, x\rangle := x^*(x)$ for each pair $(x^*, x) \in X^* \times X$. The closed ball with radius $r > 0$ and center x in X or X^* is denoted by $\mathbb{B}(x, r)$; in particular, $\mathbb{B} := \mathbb{B}(0, 1)$ stands for the closed unit ball in X or X^*. The duality mappings $\mathrm{J} : X \to X^*$ and $\mathrm{J}^* : X^* \to X$ are the set-valued mappings defined respectively by

$$\mathrm{J}(x) := \{x^* \in X^* | \langle x^*, x\rangle = \|x^*\|^2 = \|x\|^2\} \quad \text{for each } x \in X$$

and

$$\mathrm{J}^*(x^*) := \{x \in X | \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\} \quad \text{for each } x^* \in X^*.$$

Clearly, X is reflexive if and only if J is surjective, and in particular, when X is a Hilbert space, then one has that $\mathrm{J} = \mathrm{J}^* = \mathbf{I}$, the identity on X; see for example [3, 11, 39].

For a nonempty set A in X or X^*, the closure (resp. interior, weak*-closure, convex hull, convex conical hull, polar) of A is denoted by $\mathrm{cl}A$ (resp. intA, cl^*A, $\mathrm{co}A$, $\mathrm{con}A$, A°). As usual, we use $\mathrm{N}_A(\cdot)$ and $\mathrm{T}_A(\cdot)$ to stand for the norm cone mapping and the tangent cone mapping of A (assuming that A is closed and convex), defined by

$$\mathrm{N}_A(x) := \{x^* \in X^* | \langle x^*, y - x\rangle \le 0, \ \forall y \in A\} \quad \text{for each } x \in A$$

and

$$\mathrm{T}_A(x) := (\mathrm{N}_A(x))^\circ = \mathrm{cl}\,\mathrm{con}(A - x) \quad \text{for each } x \in A,$$

respectively. Associated to A, the distance function and the metric projection operator are denoted by $\mathrm{d}_A(\cdot)$ and $\mathrm{P}(\cdot|A)$, and defined by

$$\mathrm{d}_A(x) := \inf_{z \in A} \|z - x\| \quad \text{for each } x \in X$$

and

$$\mathrm{P}(x|A) := \{y \in A | \|x - y\| = \mathrm{d}_A(x)\} \quad \text{for each } x \in X,$$

respectively. Obviously, d_A is Lipschitz continuous on X with modulus 1, and it is also well-known (see, e.g., [39]) that, in the case when A is nonempty closed and convex, A is proximal (i.e., $\mathrm{P}(x|A) \neq \emptyset$ for each $x \in X$) if X is a reflexive space, and $\mathrm{P}(\cdot|A)$ is Lipschitz continuous with modulus 1 if X is a Hilbert space. Some other useful properties are described in the following lemma; see [39, Theorem 3.8.4 and Corollary 3.8.5] for (i) and (ii), and [5, Theorem A.1] for (iii).

LEMMA 2.1. *Let $A \subseteq X$ be a nonempty closed and convex set, and let $\bar{x} \in A$. Then the following assertions hold:*

(i) *For any $x \in X$, $\bar{x} \in \mathrm{P}(x|A)$ if and only if $\mathrm{J}(x - \bar{x}) \cap \mathrm{N}_A(\bar{x}) \neq \emptyset$.*
(ii) *For any $v \in \mathrm{J}^*\mathrm{N}_A(\bar{x})$ and $t > 0$, $\bar{x} \in \mathrm{P}(\bar{x} + tv|A)$.*
(iii) *For any $x \in X$, $\mathrm{d}_A(x) = \sup_{y \in A} \mathrm{d}_{y+\mathrm{T}_A(y)}(x)$.*

For $A \subseteq X$, the indicator function and the support function of A are defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ +\infty, & x \in X \setminus A \end{cases}$$

and

$$\delta_A^*(x^*) := \sup\{\langle x^*, x\rangle | x \in A\} \quad \text{for each } x^* \in X^*,$$

respectively. The following lemma is taken from [5, Theorem A.1], which will be used frequently in what follows.

LEMMA 2.2. *Let E and F be two nonempty convex subsets of X^*, and let C be a nonempty closed and convex cone in X. Then we have that*

$$\mathrm{d}_C(x) = \delta^*_{C^\circ \cap \mathbb{B}}(x) \quad \textit{for each } x \in X \tag{2.1}$$

and

$$[\delta^*_E(x) \le \delta^*_F(x),\ \forall x \in C] \Leftrightarrow [\delta^*_E(x) \le \delta^*_{F+C^\circ}(x),\ \forall x \in X] \Leftrightarrow E \subseteq \mathrm{cl}^*(F + C^\circ). \tag{2.2}$$

Another useful lemma regarding the distance function and metric projection is as follows.

LEMMA 2.3. *Let $0 < r \le +\infty$ and S_0 be a nonempty closed set. Let $x_0 \in S_0$ and $x \in \mathbb{B}(x_0, r)$. Then we have that*

$$\mathrm{d}_{S_0}(x) = \mathrm{d}_{S_0 \cap \mathbb{B}(x_0, 2r)}(x) \quad \textit{and} \quad \mathrm{P}(x|S_0) \subseteq \mathbb{B}(x_0, 2r). \tag{2.3}$$

Proof. Let $y \in S_0 \setminus \mathbb{B}(x_0, 2r)$. Then one has that $2\|x - x_0\| < \|y - x_0\|$ (noting that $\|x - x_0\| \le r$ and $\|y - x_0\| > 2r$), and so

$$\mathrm{d}_{S_0 \cap \mathbb{B}(x_0, 2r)}(x) \le \|x_0 - x\| < \|y - x_0\| - \|x_0 - x\| \le \|y - x\|.$$

Hence (2.3) holds, and the proof is complete. □

Let $f : X \to \overline{\mathbb{R}}$ be a proper and convex function. As usual, the effective domain and the epigraph of f are denoted by $\mathrm{dom} f$ and $\mathrm{epi} f$, and defined by

$$\mathrm{dom} f := \{x \in X | f(x) < +\infty\}$$

and

$$\text{epi}f := \{(x,r)|f(x) \leq r\},$$

respectively. Recall that f is lsc on X if and only if its epigraph epi$f$ is closed in $X \times \overline{\mathbb{R}}$, and its lsc hull (or its closure) is the function cl$f : X \to \overline{\mathbb{R}}$ satisfying

$$\text{epi}(\text{cl}f) = \text{cl}(\text{epi}f),$$

which is the greatest lsc function not exceeding f. We recall in the following definition the notions of the subdifferential and the ε-subdifferential for a proper and convex function.

DEFINITION 2.4. *Let $f : X \to \overline{\mathbb{R}}$ be a proper and convex function, and let $\varepsilon \geq 0$ and $\bar{x} \in \text{dom}f$. The ε-subdifferential of f at $\bar{x}$ is defined by*

$$\partial_\varepsilon f(\bar{x}) := \{x^* \in X^* | \langle x^*, x - \bar{x}\rangle \leq f(x) - f(\bar{x}) + \varepsilon,\ \forall x \in X\}.$$

In particular, $\partial f(\bar{x}) := \partial_0 f(\bar{x})$ is called the subdifferential of f at $\bar{x}$.

REMARK 2.5. (a) *Recall that the directional derivative $f'(\bar{x};\cdot) : X \to \overline{\mathbb{R}}$ of f at $\bar{x}$ is defined by*

$$f'(\bar{x};w) = \lim_{t\downarrow 0} \frac{f(\bar{x}+tw) - f(\bar{x})}{t} = \inf_{t>0} \frac{f(\bar{x}+tw) - f(\bar{x})}{t} \quad \textit{for each } w \in X.$$

Then, by [39, Corollary 2.4.15], one has the following relationship between $f'(\bar{x};\cdot)$ and $\partial f(\bar{x})$:

$$\text{cl}f'(\bar{x};\cdot) = \delta^*_{\partial f(\bar{x})}(\cdot). \tag{2.4}$$

(b) *In particular, $\mathrm{J}(x) = \partial(\frac{1}{2}\|\cdot\|^2)(x)$ for each $x \in X$ and*

$$\partial \mathrm{d}_A(x) = \mathrm{N}_A(x) \cap \mathbb{B} \quad \textit{and} \quad \partial\delta_A(x) = \mathrm{N}_A(x) \quad \textit{for each } x \in A, \tag{2.5}$$

where $A \subseteq X$ is a nonempty closed and convex set.

The sum rules for subdifferentials and ε-subdifferentials are given in the following lemma, which are known in [39, Corollary 2.6.7 and Theorem 2.8.7] and will be used in section 4.

LEMMA 2.6. *Let $f_1, f_2 : X \to \overline{\mathbb{R}}$ be two proper lsc and convex functions. Let $z \in \text{dom}f_1 \cap \text{dom}f_2$ and $\varepsilon \geq 0$. Then the following equalities are true:*

(i) $\partial_\varepsilon(f_1+f_2)(z) = \cap_{\eta>0}\text{cl}^*(\cup_{\varepsilon_i\geq 0, \varepsilon+\eta=\varepsilon_1+\varepsilon_2}(\partial_{\varepsilon_1}f_1(z) + \partial_{\varepsilon_2}f_2(z)))$.

(ii) $\partial(f_1+f_2)(z) = \cap_{\eta>0}\text{cl}^*(\partial_\eta f_1(z) + \partial_\eta f_2(z))$.

Further, suppose that f_1 is continuous at some point $\bar{x} \in \text{dom}f_2$. Then the following equalities are true:

(iii) $\partial_\varepsilon(f_1+f_2)(z) = \cup_{\varepsilon_i\geq 0, \varepsilon=\varepsilon_1+\varepsilon_2}(\partial_{\varepsilon_1}f_1(z) + \partial_{\varepsilon_2}f_2(z))$.

(iv) $\partial(f_1+f_2)(z) = \partial f_1(z) + \partial f_2(z)$.

3. Weak sharp minima for convex optimization problems. Let X be a normed space and $f : X \to \overline{\mathbb{R}}$ be a proper lsc and convex function. In this section, we consider the following convex optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in X, \end{array} \tag{3.1}$$

denoting its solution set by $\bar{S}$, i.e., $\bar{S} = \operatorname{argmin}_{x\in X} f(x)$. Throughout the section, we define the set-valued mapping $\mathrm{L}_f : \mathbb{R} \to 2^X$ by

$$\mathrm{L}_f(\lambda) := \{x \in X | f(x) = \lambda\} \qquad \text{for each } \lambda \in \mathbb{R},$$

and make the following assumption:

- $\lambda \in \mathbb{R}$ and $S_0 \subseteq \mathrm{L}_f(\lambda)$ is a nonempty closed and convex set.

The notions of weak sharp minima in the following definition are the extensions of the corresponding ones in [5] for the special case when $S_0 = \bar{S}$.

Definition 3.1. *Let $\alpha > 0$ and $x_0 \in S_0$.*

(a) x_0 is called a local weak sharp minimum over S_0 for f with modulus α if there exists $r > 0$ such that

$$\alpha \mathrm{d}_{S_0}(x) \le f(x) - f(x_0) \quad \text{for each } x \in \mathbb{B}(x_0, r). \tag{3.2}$$

(b) S_0 is called a set of boundedly weak sharp minima for f if, for each $r > 0$, there exists $\alpha (:= \alpha_r) > 0$ such that (3.2) holds with 0 in place of x_0.

(c) S_0 is called a set of (global) weak sharp minima for f with modulus α if (3.2) holds for $r = +\infty$.

Let $0 < r \le +\infty$. Then the following implications are clear by definition:

$$(3.2) \Rightarrow [\mathrm{L}_f(\lambda) \cap \mathbb{B}(x_0, r) = S_0 \cap \mathbb{B}(x_0, r) = \bar{S} \cap \mathbb{B}(x_0, r)]; \tag{3.3}$$

consequently

$$[\text{(c) in Definition 3.1}] \Rightarrow [\text{(b) in Definition 3.1}] \Rightarrow [S_0 = \mathrm{L}_f(\lambda) = \bar{S}]. \tag{3.4}$$

Remark 3.2. *Let $S \subseteq X$ be a nonempty closed and convex set. Associated to the following constrained optimization problem*

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S, \end{array} \tag{3.5}$$

we define

$$f_S(x) := (f + \delta_S)(x) = \begin{cases} f(x), & x \in S, \\ +\infty, & \textit{otherwise}. \end{cases}$$

Then the constrained optimization problem (3.5) *is equivalent to problem* (3.1) *with f_S in place of f, and $\mathrm{L}_{f_S}(\lambda) = S \cap \mathrm{L}_f(\lambda)$ for each $\lambda \in \mathbb{R}$. Moreover, letting $S_0 \subseteq \mathrm{L}_{f_S}(\lambda)$, one has that S_0*

is a set of global (or boundedly) weak sharp minima for f over the set S (see [5] for the special case when $S_0 = \text{argmin}_{x\in S} f$) if and only if S_0 is a set of global (or boundedly) weak sharp minima for f_S. Similar result holds for the local weak sharp minima.

In the special case when $S_0 = \bar{S}$, the assertions in the following proposition on the characterizations for the weak sharp minima are known in [5, Theorems 2.3, 5.2 and 6.3], which will also be used for our study in sequel.

PROPOSITION 3.3. *Let $\alpha > 0$ and $x_0 \in \bar{S}$. Then we have the following assertions:*

(i) *If x_0 is a local weak sharp minimum for f (over $\bar{S}$) with modulus α, then there exists $r > 0$ such that*

$$\alpha\mathbb{B} \cap \mathrm{N}_{\bar{S}}(z) \subseteq \partial f(z) \quad \text{for each } z \in \bar{S} \cap \mathbb{B}(x_0, r). \tag{3.6}$$

(ii) *If $\bar{S}$ is a set of boundedly weak sharp minima for f, then, for each $r > 0$, there exists $\alpha(:= \alpha_r) > 0$ such that (3.6) holds (with 0 in place of x_0).*

(iii) *$\bar{S}$ is a set of weak sharp minima for f with modulus α if and only if (3.6) holds for $r = +\infty$.*

The following two propositions are the key tools for our study in this section.

PROPOSITION 3.4. *Let $S_0 \subseteq \text{dom} f$ be nonempty closed and convex, and let $\alpha > 0$ and $z \in S_0$. Consider the following statements:*

$$f'(z;\nu) \geq \alpha \mathrm{d}_{\mathrm{T}_{S_0}(z)}(\nu) \quad \text{for each } \nu \in X. \tag{3.7}$$

$$\alpha\mathbb{B} \cap \mathrm{N}_{S_0}(z) \subseteq \partial f(z). \tag{3.8}$$

$$\alpha\mathbb{B} \cap \mathrm{N}_{S_0}(z) \subseteq \partial f(z) + [\mathrm{J}^*\mathrm{N}_{S_0}(z)]^\circ. \tag{3.9}$$

$$f'(z;\nu) \geq \alpha\|\nu\| \quad \text{for each } \nu \in \mathrm{J}^*\mathrm{N}_{S_0}(z). \tag{3.10}$$

$$\alpha\mathbb{B} \subseteq \mathrm{cl}^*(\partial f(z) + [\mathrm{J}^*\mathrm{N}_{S_0}(z)]^\circ). \tag{3.11}$$

Then one has that

$$(3.7) \Leftrightarrow (3.8) \Rightarrow (3.9) \Rightarrow (3.10), \tag{3.12}$$

and, if X is a Hilbert space,

$$(3.9) \Rightarrow (3.11) \Rightarrow (3.10). \tag{3.13}$$

Proof. We first note by (2.1) in Lemma 2.2 (applied to $\mathrm{T}_{S_0}(z)$ in place of C) that

$$\alpha \mathrm{d}_{\mathrm{T}_{S_0}(z)}(\nu) = \alpha \delta^*_{\mathbb{B}\cap[\mathrm{T}_{S_0}(z)]^\circ}(\nu) = \delta^*_{\alpha\mathbb{B}\cap \mathrm{N}_{S_0}(z)}(\nu) \quad \text{for each } \nu \in X.$$

It follows from (2.4) and (2.2) (applied to $\alpha\mathbb{B}\cap \mathrm{N}_{S_0}(z)$, $\partial f(z)$, X in place of E, F, C) that

$$\begin{aligned} (3.7) \quad &\Leftrightarrow \quad \mathrm{cl} f'(z;\nu) \geq \alpha \mathrm{d}_{\mathrm{T}_{S_0}(z)}(\nu) \quad \text{for each } \nu \in X \\ &\Leftrightarrow \quad \delta^*_{\partial f(z)}(\nu) \geq \delta^*_{\alpha\mathbb{B}\cap \mathrm{N}_{S_0}(z)}(\nu) \quad \text{for each } \nu \in X \\ &\Leftrightarrow \quad (3.8), \end{aligned}$$

where the first equivalence holds because the function $\mathrm{d}_{\mathrm{T}_{S_0}(z)}(\cdot)$ is continuous, and (3.7) $\Leftrightarrow$ (3.8) is shown. Hence, to complete the proof of (3.12), it suffices to verify the implication (3.9) $\Rightarrow$ (3.10) because the implication (3.8) $\Rightarrow$ (3.9) is trivial. Consider the following statement:

$$\mathrm{cl} f'(z;\cdot)(\nu) \geq \alpha\|\nu\| \quad \text{for each } \nu \in \mathrm{J}^*\mathrm{N}_{S_0}(z). \tag{3.14}$$

Clearly, to show the implication (3.9) $\Rightarrow$ (3.10), we only need to check the following implication:

$$(3.9) \Rightarrow (3.14). \tag{3.15}$$

To do this, we note that

$$\delta^*_{\alpha\mathbb{B}^*}(\nu) = \delta^*_{\alpha\mathbb{B}^*\cap \mathrm{N}_{S_0}(z)}(\nu) = \alpha\|\nu\| \quad \text{for each } \nu \in \mathrm{J}^*\mathrm{N}_{S_0}(z) \tag{3.16}$$

because, for each $\nu \in \mathrm{J}^*\mathrm{N}_{S_0}(z)$, one has that

$$\|\nu\| = \delta^*_{\mathbb{B}^*}(\nu) \geq \delta^*_{\mathbb{B}^*\cap \mathrm{N}_{S_0}(z)}(\nu) \geq \langle \frac{u^*}{\|u^*\|}, \nu\rangle = \|\nu\|,$$

where $u^* \in \mathrm{N}_{S_0}(z)$ is such that $\nu \in \mathrm{J}^*(u^*)$. To proceed, suppose that (3.9) holds. Then one has by definition that

$$\delta^*_{\alpha\mathbb{B}^*\cap \mathrm{N}_{S_0}(z)}(\nu) \leq \delta^*_{\partial f(z)+[\mathrm{J}^*\mathrm{N}_{S_0}(z)]^\circ}(\nu) \quad \text{for each } \nu \in X,$$

and, in particular,

$$\delta^*_{\alpha\mathbb{B}^*\cap \mathrm{N}_{S_0}(z)}(\nu) \leq \delta^*_{\partial f(z)}(\nu) \quad \text{for each } \nu \in \mathrm{J}^*\mathrm{N}_{S_0}(z). \tag{3.17}$$

Thus, combining (3.16) and (3.17), we apply (2.4) to conclude that

$$\mathrm{cl} f'(z;\cdot)(\nu) = \delta^*_{\partial f(z)}(\nu) \geq \alpha\|\nu\| \quad \text{for each } \nu \in \mathrm{J}^*\mathrm{N}_{S_0}(z).$$

Hence, the implication (3.15) is checked, and the proof for (3.12) is complete.

Finally, assume that X is a Hilbert space. Then $\mathrm{J}^* = \mathbf{I}$ and so $\mathrm{J}^*\mathrm{N}_{S_0}(z) = \mathrm{N}_{S_0}(z)$ is a convex cone. Thus, one applies (2.2) (to $\alpha\mathbb{B}$, $\partial f(z)$, $\mathrm{N}_{S_0}(z)$ in place of E, F, C) again to get that

$$(3.11) \quad \Leftrightarrow \quad [\delta^*_{\alpha\mathbb{B}}(\nu) \leq \delta^*_{\partial f(z)}(\nu),\ \forall \nu \in \mathrm{N}_{S_0}(z)].$$

This, together with (3.16) and (2.4), implies that (3.11) $\Leftrightarrow$ (3.14). Noting clearly that (3.14) $\Rightarrow$ (3.10), we see that (3.13) holds by (3.15). The proof is complete. □

Proposition 3.5. *Let $\alpha > 0$, $0 < r \le +\infty$ and $x_0 \in S_0$. Then we have the following assertion:*

(i) *If* (3.10) *holds for each $z \in S_0 \cap \mathbb{B}(x_0, 2r)$, then, for each $x \in \mathbb{B}(x_0, r)$, one has*

$$f'(z; x - z) \ge \alpha \mathrm{d}_{S_0}(x) \quad \text{for each } z \in \mathrm{P}(x|S_0). \tag{3.18}$$

(ii) *If* (3.10) *holds for each $z \in S_0 \cap \mathbb{B}(x_0, 2r)$ and S_0 is proximal, then* (3.2) *holds.*

(iii) *If* (3.8) *holds for each $z \in S_0 \cap \mathbb{B}(x_0, 2r)$, then* (3.2) *holds but with $\frac{\alpha}{2}$ in place of α in the case when $r < +\infty$.*

Proof. (i) Let $x \in \mathbb{B}(x_0, r)$ and $z \in \mathrm{P}(x|S_0)$. Then one has by Lemma 2.3 that $z \in \mathbb{B}(x_0, 2r)$, and so $z \in S_0 \cap \mathbb{B}(x_0, 2r)$. Furthermore, by Lemma 2.1(i), one sees that $\mathrm{J}(x - z) \cap \mathrm{N}_{S_0}(z) \ne \emptyset$, and can take $z^* \in \mathrm{J}(x - z) \cap \mathrm{N}_{S_0}(z)$. Thus $x - z \in \mathrm{J}^*(z^*)$ by definition and so $x - z \in \mathrm{J}^*\mathrm{N}_{S_0}(z)$. Consequently, (3.10) is applicable and then

$$f'(z; x - z) \ge \alpha \|x - z\| = \alpha \mathrm{d}_{S_0}(x)$$

(noting $z \in \mathrm{P}(x|S_0)$). Hence (3.18) is proved.

(ii) Assume that (3.10) holds for each $z \in S_0 \cap \mathbb{B}(x_0, 2r)$ and S_0 is proximal. Then, for each $x \in \mathbb{B}(x_0, r)$, one can chose $z \in \mathrm{P}(x|S_0)$ and so (3.18) holds by the established assertion (i). Furthermore, we note by the convexity of f that

$$f(x) - f(x_0) = f(x) - f(z) \ge f'(z; x - z) \quad \text{for each } x \in X. \tag{3.19}$$

This, together with (3.18), implies that (3.2) holds, and the proof for (ii) is complete.

(iii) Assume that (3.8) holds for each $z \in S_0 \cap \mathbb{B}(x_0, 2r)$ and let $x \in \mathbb{B}(x_0, r)$. Then, by Proposition 3.4, (3.7) holds for each $z \in S_0 \cap \mathbb{B}(x_0, 2r)$. This, together with (3.19), entails that

$$f(x) - f(x_0) \ge \alpha \mathrm{d}_{\mathrm{T}_{S_0}(z)}(x - z) \quad \text{for each } z \in S_0 \cap \mathbb{B}(x_0, 2r). \tag{3.20}$$

Thus, if $r = +\infty$, then (3.2) follows from Lemma 2.1(iii) (applied to S_0 in place of A). It remains to consider the case when $r < +\infty$. By (3.20), it suffices to prove that

$$\mathrm{d}_{S_0}(x) \le 2 \sup_{z \in S_0 \cap \mathbb{B}(x_0, 2r)} \mathrm{d}_{\mathrm{T}_{S_0}(z)}(x - z).$$

For this purpose, we below verify that

$$\mathbb{B}^* \cap \mathrm{N}_{S_0 \cap \mathbb{B}(x_0, 2r)}(z) \subseteq 2\mathbb{B}^* \cap \mathrm{N}_{S_0}(z) + N_{\mathbb{B}(x_0, 2r)}(z) \quad \text{for each } z \in S_0 \cap \mathbb{B}(x_0, 2r). \tag{3.21}$$

Granting this, one has by (3.8) that, for each $z \in \tilde{S}_0$,

$$\frac{\alpha}{2} \mathbb{B}^* \cap \mathrm{N}_{S_0 \cap \mathbb{B}(x_0, 2r)}(z) \subseteq \partial f(z) + N_{\mathbb{B}(x_0, 2r)}(z) = \partial \tilde{f}(z),$$

where $\tilde{f} : X \to \overline{\mathbb{R}}$ is the function defined by $\tilde{f} := f + \delta_{\mathbb{B}(x_0, 2r)}$ and $\tilde{S}_0 := S_0 \cap \mathbb{B}(x_0, 2r) \subseteq \mathrm{L}_{\tilde{f}}(\lambda)$; hence (3.8) holds for each $z \in \tilde{S}_0$ with $\frac{\alpha}{2}$, $\tilde{f}$, $\tilde{S}_0$ in place of α, f, S_0. Thus, applying the conclusion (to $\tilde{f}$, $\tilde{S}_0$ in place of f, S_0) just established for the case when $r = +\infty$, we have that

$$f(x) - f(x_0) = \tilde{f}(x) - \tilde{f}(x_0) \ge \frac{\alpha}{2} \mathrm{d}_{\tilde{S}_0}(x) = \frac{\alpha}{2} \mathrm{d}_{S_0}(x),$$

where the last equality is because of (2.3), and so (3.2) holds with $\frac{\alpha}{2}$ in place of α.

To proceed, we note by the equivalence in Proposition 3.3(iii) (applied to $\tilde{S}_0$, $2\mathrm{d}_{S_0}(\cdot)+\delta_{\mathbb{B}(x_0,2r)}(\cdot)$ in place of S_0, $f(\cdot)$) that, thanks to (2.5), (3.21) is equivalent to

$$\mathrm{d}_{\tilde{S}_0}(x)\leq 2\mathrm{d}_{S_0}(x)\quad\text{for each } x\in\mathbb{B}(x_0,2r). \tag{3.22}$$

Therefore, to complete the proof, we have to show (3.22). To do this, let $x\in\mathbb{B}(x_0,2r)$. Without loss of generality, we assume that $x_0=0$. Set $z:=\frac{2r\mathrm{P}(x|S_0)}{\mathrm{d}_{S_0}(x)+2r}$. Then

$$\|z\|=\frac{2r\|\mathrm{P}(x|S_0)-x\|+2r\|x\|}{\mathrm{d}_{S_0}(x)+2r}\leq\frac{2r\mathrm{d}_{S_0}(x)+4r^2}{\mathrm{d}_{S_0}(x)+2r}=2r,$$

and so $z\in S_0\cap 2r\mathbb{B}=\tilde{S}_0$ (as $z\in S_0$ is clear). Thus,

$$\mathrm{d}_{\tilde{S}_0}(x)\leq\|x-z\|=\frac{\|\mathrm{d}_{S_0}(x)\,x+2r(x-\mathrm{P}(x|S_0)\|}{\mathrm{d}_{S_0}(x)+2r}\leq\frac{4r\mathrm{d}_{S_0}(x)}{\mathrm{d}_{S_0}(x)+2r}\leq 2\mathrm{d}_{S_0}(x),$$

as desired to show, and the proof is complete. □

Our first theorem in this section, which extends/improves the corresponding ones in [5, 39] for the special case when $S_0=\bar{S}$ (noting the equivalence between the constrained optimization problem (3.5) and the unconstrained optimization problem (3.1) explained in Remark 3.2), is as follows. In particular, in the special case when $S_0=\bar{S}$, the equivalence between statements (i)-(iii) were established in [5, Theorem 2.3]; the implications (i) ⇒ (vi) ⇒ (v) were provided in [39, Theorem 3.10.1], where the equivalence between statements (i)-(vi) was also established for the case when X is reflexive; while the equivalence between statements (i)-(iii) and statements (v)-(vii) were proved in [5, Theorem 2.3] for the case when X is a Hilbert space.

Theorem 3.6. *Let $\alpha>0$. Then we have the following assertions.*

(I) *The following statements are equivalent:*

(i) *S_0 is a set of (global) weak sharp minima for f with modulus α.*

(ii) *For each $z\in S_0$, (3.7) holds:*

$$f'(z;\nu)\geq\alpha\mathrm{d}_{\mathrm{T}_{S_0}(z)}(\nu)\quad\textit{for each } \nu\in X.$$

(iii) *For each $z\in S_0$, (3.8) holds:*

$$\alpha\mathbb{B}\cap\mathrm{N}_{S_0}(z)\subseteq\partial f(z).$$

(II) *If S_0 is proximal (e.g., X is reflexive), then each of* (i)-(iii) *is equivalent to each of the following statements:*

(iv) *For each $z\in S_0$, (3.9) holds:*

$$\alpha\mathbb{B}\cap\mathrm{N}_{S_0}(z)\subseteq\partial f(z)+[\mathrm{J}^*\mathrm{N}_{S_0}(z)]^\circ.$$

(v) *For each $z\in S_0$, (3.10) holds:*

$$f'(z;\nu)\geq\alpha\|\nu\|\quad\textit{for each } \nu\in\mathrm{J}^*\mathrm{N}_{S_0}(z).$$

(vi) *For any* $x \in X$ *and* $z \in \mathrm{P}(x|S_0)$,

$$f'(z; x - z) \geq \alpha \mathrm{d}_{S_0}(x). \tag{3.23}$$

(III) *If* X *is a Hilbert space, then each of* (i)-(vi) *is equivalent to each of the following statements:*

(vii) *For each* $z \in S_0$, (3.11) *holds:*

$$\alpha \mathbb{B} \subseteq \mathrm{cl}(\partial f(z) + [\mathrm{N}_{S_0}(z)]^\circ). \tag{3.24}$$

(viii) *For each* $z \in S_0$,

$$\hat{\alpha} \mathbb{B} \subseteq \partial f(z) + [\mathrm{N}_{S_0}(z)]^\circ \quad \text{for each } \hat{\alpha} \in (0, \alpha). \tag{3.25}$$

Proof. We first prove that

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Leftrightarrow \text{(vi)}. \tag{3.26}$$

Granting this, one completes the proof for assertion (I).

To verify (3.26), we note that the implications (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) follow from Proposition 3.4 and (v) $\Rightarrow$ (vi) from Proposition 3.5(i) (by taking $r = +\infty$). Thus it suffices to testify (i) $\Leftrightarrow$ (iii) and (vi) $\Rightarrow$ (v). To do this, we note by Proposition 3.5(iii) (by taking $r = +\infty$) that (iii)$\Rightarrow$(i). Hence by (3.4) if either (i) or (iii) holds then $S_0 = \mathrm{L}_f(\lambda) = \bar{S}$. Thus the equivalence (i) $\Leftrightarrow$ (iii) follows immediately from Proposition 3.3(iii).

To show (vi) $\Rightarrow$ (v), suppose that (vi) holds, and let $z \in S_0$. Fix $\nu \in \mathrm{J}^*\mathrm{N}_{S_0}(z)$ and write $x := z + \nu$. Then, by Lemma 2.1(ii), one has that $z \in \mathrm{P}(x|S_0)$, and so $\mathrm{d}_{S_0}(x) = \|x - z\| = \|v\|$. Thus, from (vi), it follows that

$$f'(z; v) = f'(z; x - z) \geq \alpha \mathrm{d}_{S_0}(x) = \alpha \|\nu\|.$$

This shows (3.10) as $\nu \in \mathrm{J}^*\mathrm{N}_{S_0}(z)$ is arbitrary, and so (v) is seen to hold. Therefore, the proof for (3.26) is complete.

To verify assertion (II), assume that S_0 is proximal. By (3.26), we only need to show the implication (v) $\Rightarrow$ (i). To do this, assume (v), that is (3.10) holds for each $z \in S_0$. Thus Proposition 3.5(ii) is applicable to conclude that (3.2) holds for $r = +\infty$, that is, (i) holds, and assertion (II) is proved.

Finally, we show assertion (III). For this purpose, suppose that X is a Hilbert space. We only need to verify that (iv)$\Rightarrow$(vii)$\Leftrightarrow$(viii)$\Rightarrow$(v). We will complete the proof by showing that (3.9) $\Rightarrow$ (3.24) $\Leftrightarrow$ (3.25) $\Rightarrow$ (3.10) holds for each $z \in S_0$. To do this, let $z \in S_0$. Note that (3.24) coincides with (3.11), and so (3.9) $\Rightarrow$ (3.24) $\Rightarrow$ (3.10) by (3.13) in Proposition 3.4. To show (3.24)$\Leftrightarrow$(3.25), one notes that

$$\mathrm{int}(\mathrm{cl}(\partial f(z) + [\mathrm{N}_{S_0}(z)]^\circ)) = \mathrm{int}(\partial f(z) + [\mathrm{N}_{S_0}(z)]^\circ).$$

It follows that

$$(3.24) \Leftrightarrow [\mathrm{int}(\alpha \mathbb{B}) \subseteq \mathrm{int}(\partial f(z) + [\mathrm{N}_{S_0}(z)]^\circ)] \Leftrightarrow (3.25).$$

The proof is complete. $\square$

REMARK 3.7. *As argued in the proof, the following equivalences/implications for statements (i)-(vi) given in Theorem 3.6 hold in general normed linear spaces (see* (3.26)*):*

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Leftrightarrow \text{(vi)}.$$

Theorems 3.8 and 3.9 below provide the analogues of Theorem 3.6 for the local weak sharp minima over S_0 and the boundedly weak sharp minima, respectively. As described in Proposition 3.3, for the special case when $S_0 = \bar{S}$, the implication (i)⇒(ii) (for statements (i) and (ii) given in Theorem 3.8 or Theorem 3.9) was showed in [5, Theorems 5.2 and 6.3], where the converse implication was proved only for the case when X is a Hilbert space or a finite-dimensional space. Moreover, in the case when S_0 is complete, the implication (ii)⇒(i) was also known as a direct consequence of [43, Propositon 4.2]. However, the implication (ii)⇒(i) for general linear normed spaces and other equivalent characterizations for the local weak sharp minima or the boundedly weak sharp minima seem new, even for the case when $S_0 = \bar{S}$ or/and when X is a reflexive space or a Hilbert space.

THEOREM 3.8. *Let $x_0 \in S_0$ and $\alpha > 0$. Then we have the following assertions.*

(I) *The following statements are equivalent but* (iii) ⇒ (i) *for possible different values of $\alpha > 0$:*
- (i) *x_0 is a local weak sharp minima over S_0 for f with modulus α.*
- (ii) *There exists $r > 0$ such that* (3.7) *holds for each $z \in S_0 \cap \mathbb{B}(x_0, r)$.*
- (iii) *There exists $r > 0$ such that* (3.8) *holds for each $z \in S_0 \cap \mathbb{B}(x_0, r)$.*

(II) *If S_0 is proximal, then each of* (i)-(iii) *is equivalent to each of the following statements:*
- (iv) *There exists $r > 0$ such that* (3.9) *holds for each $z \in S_0 \cap \mathbb{B}(x_0, r)$.*
- (v) *There exists $r > 0$ such that* (3.10) *holds for each $z \in S_0 \cap \mathbb{B}(x_0, r)$.*
- (vi) *There exists $r > 0$ such that* (3.23) *holds for each $x \in \mathbb{B}(x_0, r)$ and each $z \in \mathrm{P}(x|S_0)$.*

(III) *If X is a Hilbert space, then each of* (i)-(vi) *is equivalent to each of the following statements:*
- (vii) *There exists $r > 0$ such that* (3.24) *holds for each $z \in S_0 \cap \mathbb{B}(x_0, r)$.*
- (viii) *There exists $r > 0$ such that* (3.25) *holds for each $z \in S_0 \cap \mathbb{B}(x_0, r)$.*

Proof. The proofs for (ii) ⇔ (iii) ⇒ (iv) ⇒ (v) ⇒ (vi), and for assertions (II) and (III) are similar to these for the corresponding ones in Theorem 3.6 (by choosing an appropriate r), and so we omit them here.

To complete the proof, we only need to verify (vi) ⇒ (v) and (i) ⇔ (iii). We first show the implication (vi) ⇒ (v). To do this, assume that (vi) holds and let $r > 0$ be such that (3.23) holds for any $x \in \mathbb{B}(x_0, r)$ and $z \in \mathrm{P}(x|S_0)$. It suffices to prove that (3.10) holds for each $z \in S_0 \cap \mathbb{B}(x_0, \frac{r}{2})$. To do this, let $z \in S_0 \cap \mathbb{B}(x_0, \frac{r}{2})$ and fix $\nu \in \mathrm{J}^*\mathrm{N}_{S_0}(z)$. Write $x := z + \frac{r\nu}{2\|\nu\|}$. Then, by Lemma 2.1(ii), one has that $z \in \mathrm{P}(x|S_0)$, and so $\mathrm{d}_{S_0}(x) = \|x - z\| = \frac{r}{2}$. Since $\|x - x_0\| \le \|z - x_0\| + \frac{r}{2} < r$, (3.23) is applicable, and so

$$f'(z; \frac{r\nu}{2\|\nu\|}) = f'(z; x - z) \ge \alpha \mathrm{d}_{S_0}(x) = \frac{\alpha r}{2}.$$

Hence, $f'(z; \nu) \ge \alpha\|\nu\|$ because $f'(z; \cdot)$ is positively homogeneous. Thus, (3.10) is checked as $\nu \in \mathrm{J}^*\mathrm{N}_{S_0}(z)$ is arbitrary, and (v) is obtained as desired to show.

Below, we verify the equivalence (i) ⇔ (iii). To this end, assume first that (i) holds. Then there exists $r' > 0$ such that (3.2) holds with r' in place of r. This, together with (3.3), implies that

$$S_0 \cap \mathbb{B}(x_0, r') = \bar{S} \cap \mathbb{B}(x_0, r'). \tag{3.27}$$

Furthermore, since $S_0 \subseteq \bar{S}$, it follows from (3.2) (applied to r' in place of r) that

$$\alpha \mathrm{d}_{\bar{S}}(x) \leq f(x) - f(x_0) \quad \text{for each } x \in \mathbb{B}(x_0, r').$$

This means that x_0 is a local weak sharp minima (over $\bar{S}$) for f with modulus α. Thus, one applies Proposition 3.3(i) to conclude that there exists $r > 0$ such that (3.6) holds. Without loss of generality, assume that $r < r'$. Then, by (3.27),

$$\mathrm{N}_{\bar{S}}(z) = \mathrm{N}_{\bar{S}\cap\mathbb{B}(x_0,r')}(z) = \mathrm{N}_{S_0\cap\mathbb{B}(x_0,r')}(z) = \mathrm{N}_{S_0}(z)$$

holds for each $z \in S_0 \cap \mathbb{B}(x_0, r)$, and so it follows from (3.6) that

$$\alpha\mathbb{B} \cap \mathrm{N}_{S_0}(z) \subseteq \partial f(z) \quad \text{for each } z \in S_0 \cap \mathbb{B}(x_0, r).$$

Thus (iii) is checked.

Conversely, assume that (iii) holds, that is there exists $r > 0$ such that (3.8) holds for each $z \in S_0 \cap \mathbb{B}(x_0, 2r)$. Then, (3.2) holds by Proposition 3.5(iii) with $\frac{\alpha}{2}$ in place of α. This means that (i) holds with $\frac{\alpha}{2}$ in place of α, and the proof is complete. □

THEOREM 3.9. *The following assertions are true.*

(I) *The following statements are equivalent:*

(i) S_0 *is a set of boundedly weak sharp minima for* f*.*

(ii) *For each* $r > 0$*, there exists* $\alpha > 0$ *such that* (3.7) *holds for each* $z \in S_0 \cap r\mathbb{B}$*.*

(iii) *For each* $r > 0$*, there exists* $\alpha > 0$ *such that* (3.8) *holds for each* $z \in S_0 \cap r\mathbb{B}$*.*

(II) *If* S_0 *is proximal, then each of* (i)-(iii) *is equivalent to each of the following statements:*

(iv) *For each* $r > 0$*, there exists* $\alpha > 0$ *such that* (3.9) *holds for each* $z \in S_0 \cap r\mathbb{B}$*.*

(v) *For each* $r > 0$*, there exists* $\alpha > 0$ *such that* (3.10) *holds for each* $z \in S_0 \cap r\mathbb{B}$*.*

(vi) *For each* $r > 0$*, there exists* $\alpha > 0$ *such that* (3.23) *holds for each* $x \in r\mathbb{B}$ *and each* $z \in \mathrm{P}(x|S_0)$*.*

(III) *If* X *is a Hilbert space, then each of* (i)-(vi) *is equivalent to the following statement:*

(vii) *For each* $r > 0$*, there exists* $\alpha > 0$ *such that*

$$\alpha\mathbb{B} \subseteq \partial f(z) + [\mathrm{N}_{S_0}(z)]^{\circ} \quad \textit{for each } z \in S_0 \cap r\mathbb{B}.$$

Proof. It is similar to that for Theorem 3.8. □

REMARK 3.10. *As remarked for Theorem 3.6, for statements (i)-(vi) given in Theorem 3.8 or Theorem 3.9, implications/equivalences* (3.26) *hold in general normed linear spaces.*

4. Weak sharp minima for convex infinite optimization problems. As in the preceding sections, we assume that X is a normed space, and $f : X \to \overline{\mathbb{R}}$ is a proper lsc and convex function. This section is devoted to investigating the characterization problem of weak sharp minima for the following convex infinite optimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \Omega, \\ & \phi_y(x) \leq 0, \ \ \forall y \in Y, \end{array} \tag{4.1}$$

where Y is an index set, $\Omega \subseteq X$ is a nonempty closed and convex set, and $\phi_{(\cdot)}(\cdot) : X \times Y \to \overline{\mathbb{R}}$ is such that, for each $y \in Y$, $\phi_y(\cdot)$ is lsc and convex on X. Let Z denote the set of all feasible points for optimization problem (4.1), that is,

$$Z := \{x \in \Omega | \phi_y(x) \leq 0, \ \forall y \in Y\}.$$

For avoiding the triviality, we assume, throughout this section, that

$$Z \cap \mathrm{dom} f \neq \emptyset.$$

As in [42], we write

$$\lambda := \inf_{x \in Z} f(x) \quad \text{and} \quad S := \{x \in Z | f(x) = \lambda\} = Z \cap \mathrm{L}_f(\lambda).$$

Then S is closed and convex. For a function $\phi : X \to \overline{\mathbb{R}}$, its positive part is defined by

$$\phi_+(x) := \max\{\phi(x), 0\} \quad \text{for each } x \in X.$$

We recall in the following definition the notions of weak sharp minima property for problem (4.1). In particular, items (a) and (c) are taken from [42].

Definition 4.1. *Let $\alpha > 0$ and $x_0 \in S$. Problem* (4.1) *is said to have*

(a) *a local weak sharp minimum property at x_0 in S (with modulus α) if there exists $r > 0$ such that*

$$\alpha \mathrm{d}_S(x) \leq f(x) - f(x_0) + \sup_{y \in Y} [\phi_y]_+(x) + \mathrm{d}_\Omega(x) \quad \textit{for each } x \in \mathbb{B}(x_0, r); \tag{4.2}$$

(b) *a boundedly weak sharp minima property in S if for each $r > 0$, there exists $\alpha (:= \alpha_r) > 0$ such that* (4.2) *holds with* 0 *in place of x_0;*

(c) *a global weak sharp minima property in S (with modulus α) if* (4.2) *holds for $r = +\infty$.*

Associated to problem (4.1), we define $\hat{f} : X \to \overline{\mathbb{R}}$ by

$$\hat{f}(x) := f(x) + \sup_{y \in Y} [\phi_y]_+(x) + \mathrm{d}_\Omega(x) \quad \text{for each } x \in X. \tag{4.3}$$

Then, $\hat{f}$ is lsc and convex, and $\hat{f} = f$ on S because $\sup_{y \in Y} [\phi_y]_+(x) = 0$ and $\mathrm{d}_\Omega(x) = 0$ hold for each $x \in S$. Hence, the relationship between problem (4.1) has a global (resp. local, boundedly) weak sharp minima property in S and S is a set of global (resp. local, boundedly) weak sharp minima for $\hat{f}$ is stated as follows.

Proposition 4.2. *Problem* (4.1) *has a global (resp. local, boundedly) weak sharp minima property in S if and only if S is a set of global (resp. local, boundedly) weak sharp minima for $\hat{f}$.*

To applying characterization theorems established in above section, we need to calculate the subdifferential of $\hat{f}$, which will be done in the following subsection.

4.1. Subdifferential of the associated function. For the remainder, we use Φ to denote the sup-function of the family of proper lsc and convex functions $\{\phi_y|y\in Y\}$, defined by

$$\Phi(x):=\sup_{y\in Y}\phi_y(x)\quad \text{for each } x\in X. \tag{4.4}$$

Let $\varepsilon\geq 0$ and $x\in X$. We use $Y^\varepsilon(x)$ to denote the ε-active set at x for the family $\{\phi_y|y\in Y\}$, defined by

$$Y^\varepsilon(x):=\{y\in Y|\phi_y(x)\geq\Phi(x)-\varepsilon\}.$$

The active set at x is denoted by $Y(x)$, that is,

$$Y(x):=Y^0(x)=\{y\in Y|\phi_y(x)=\Phi(x)\}.$$

We first recall in the following proposition some subdifferential formulas for the sup-function Φ, known in [18, Corollary 8] and [39, Theorem 2.4.18].

Proposition 4.3. *Let $z\in\mathrm{dom}\Phi$. Then the following assertions hold.*
(i) *We have that*

$$\partial\Phi(z)\supseteq \mathrm{cl}^*\mathrm{co}(\cup_{y\in Y(z)}\partial\phi_y(z))$$

(ii) *If* $\mathrm{ri}\,(\mathrm{dom}\Phi)\neq\emptyset$*, then*

$$\partial\Phi(z)=\cap_{\varepsilon>0}\mathrm{cl}^*(\mathrm{co}(\cup_{y\in Y^\varepsilon(z)}\partial_{\tau\varepsilon}\phi_y(z))+\mathrm{N}_{\mathrm{dom}\Phi}(z))\quad \textit{for each } \tau>0. \tag{4.5}$$

(iii) *If assumption* (B1) *holds, and each function $\phi_y(\cdot)$ is continuous at z, then*

$$\partial\Phi(z)=\mathrm{cl}^*\mathrm{co}(\cup_{y\in Y(z)}\partial\phi_y(z)). \tag{4.6}$$

Proposition 4.4. *Suppose that* $\mathrm{ri}\,(\mathrm{dom}f\cap\mathrm{dom}\Phi)\neq\emptyset$ *and let $z\in\mathrm{dom}f\cap S$. Then*

$$\partial(f+\Phi)(z)=\cap_{\varepsilon>0}\mathrm{cl}^*[\mathrm{co}(\cup_{y\in Y^\varepsilon(z)}(\partial_\varepsilon f(z)+\partial_\varepsilon\phi_y(z)))+\mathrm{N}_{\mathrm{dom}(f+\Phi)}(z)]. \tag{4.7}$$

Proof. Consider the family of proper and convex functions $\{f+\phi_y|y\in Y\}$ on X. Then one sees that

$$f+\Phi=\sup_{y\in Y}(f+\phi_y), \tag{4.8}$$

and, for each $\varepsilon\geq 0$, the ε-active set at z for the family $\{f+\phi_y|y\in Y\}$ is equal to the one for the family $\{\phi_y|y\in Y\}$, that is,

$$\{y\in Y|(f+\phi_y)(z)\geq(f+\Phi)(z)-\varepsilon\}=Y^\varepsilon(z). \tag{4.9}$$

To apply Proposition 4.3(ii), we use Lemma 2.6(i) to estimate the ε-subdifferential of $(f+\phi_y)(z)$ for each $y\in Y$ and have that

$$\begin{aligned}\partial_\varepsilon(f+\phi_y)(z)\quad&=\cap_{\eta>0}\mathrm{cl}^*(\cup_{\varepsilon_i\geq 0,\varepsilon+\eta=\varepsilon_1+\varepsilon_2}(\partial_{\varepsilon_1}f(z)+\partial_{\varepsilon_2}\phi_y(z)))\\&\subseteq\cap_{\eta>0}\mathrm{cl}^*(\partial_{\varepsilon+\eta}f(z)+\partial_{\varepsilon+\eta}\phi_y(z)).\end{aligned}$$

Thus, by Proposition 4.3(ii), one has that

$$\begin{aligned}&\partial(f+\Phi)(z)\\ &=\cap_{\varepsilon>0}\mathrm{cl}^*[\mathrm{co}(\cup_{y\in Y^\varepsilon(z)}\partial_\varepsilon(f+\phi_y)(z))+\mathrm{N}_{\mathrm{dom}(f+\Phi)}(z)]\\ &\subseteq\cap_{\varepsilon>0}\mathrm{cl}^*[\mathrm{co}(\cup_{y\in Y^\varepsilon(z)}\cap_{\eta>0}\mathrm{cl}^*(\partial_{\varepsilon+\eta}f(z)+\partial_{\varepsilon+\eta}\phi_y(z)))+\mathrm{N}_{\mathrm{dom}(f+\Phi)}(z)].\end{aligned}\tag{4.10}$$

To proceed, we note the following clear inclusions

$$\cup_{y\in T}\mathrm{cl}^*A(y)\subseteq\mathrm{cl}^*\mathrm{co}(\cup_{y\in T}A(y))\tag{4.11}$$

for a family of sets $\{A(y)|y\in T\}$ in X^* and

$$\mathrm{cl}^*A+B\subseteq\mathrm{cl}^*(A+B)\tag{4.12}$$

for two sets A and B in X^*. Thus, one concludes by (4.11) (applied to $\{\partial_{\varepsilon+\eta}f(z)+\partial_{\varepsilon+\eta}\phi_y(z)|y\in Y^\varepsilon(z)\}$ in place of $\{A(y)|y\in T\}$) that the following inclusion holds for each $\eta>0$ and $\varepsilon>0$

$$\mathrm{co}(\cup_{y\in Y^\varepsilon(z)}\cap_{\eta>0}\mathrm{cl}^*(\partial_{\varepsilon+\eta}f(z)+\partial_{\varepsilon+\eta}\phi_y(z)))\subseteq\mathrm{cl}^*[\mathrm{co}(\cup_{y\in Y^\varepsilon(z)}(\partial_{\varepsilon+\eta}f(z)+\partial_{\varepsilon+\eta}\phi_y(z)))].$$

Combining this and (4.10), and thanks to (4.12) (applied to $\mathrm{co}(\cup_{y\in Y^\varepsilon(z)}(\partial_{\varepsilon+\eta}f(z)+\partial_{\varepsilon+\eta}\phi_y(z)))$, $\mathrm{N}_{\mathrm{dom}(f+\Phi)}(z)$ in place of A, B), we obtain that

$$\partial(f+\Phi)(z)\subseteq\cap_{\varepsilon>0}\cap_{\eta>0}\mathrm{cl}^*[\mathrm{co}(\cup_{y\in Y^\varepsilon(z)}(\partial_{\varepsilon+\eta}f(z)+\partial_{\varepsilon+\eta}\phi_y(z)))+\mathrm{N}_{\mathrm{dom}(f+\Phi)}(z)].$$

Since $Y^\varepsilon(z)\subseteq Y^{\varepsilon+\eta}(z)$ and $\partial_\varepsilon f(z)+\partial_\varepsilon\phi_y(z)\subseteq\partial_{2\varepsilon}(f+\phi_y)(z)$, it follows that

$$\begin{aligned}\partial(f+\Phi)(z)\quad&\subseteq\cap_{\varepsilon>0}\mathrm{cl}^*[\mathrm{co}(\cup_{y\in Y^\varepsilon(z)}(\partial_\varepsilon(f(z)+\partial_\varepsilon\phi_y(z)))+\mathrm{N}_{\mathrm{dom}(f+\Phi)}(z)]\\ &\subseteq\cap_{\varepsilon>0}\mathrm{cl}^*[\mathrm{co}(\cup_{y\in Y^\varepsilon(z)}\partial_{2\varepsilon}(f+\phi_y)(z))+\mathrm{N}_{\mathrm{dom}(f+\Phi)}(z)]\\ &=\partial(f+\Phi)(z),\end{aligned}$$

where the equality is because of (4.5) (applied to 2, $\{f+\phi_y|y\in Y\}$ in place of τ, $\{\phi_y|y\in Y\}$, and thanks to (4.8) and (4.9)). Therefore, (4.7) is proved and the proof is complete. □

For the sake of simplicity, we introduce the notation $E(\varepsilon,z)$ for any $z\in X$ and $\varepsilon>0$ defined by

$$E(\varepsilon,z):=\mathrm{cl}^*\left(\mathrm{co}[\cup_{y\in Y_+^\varepsilon(z)}(\partial_\varepsilon f(z)+\partial_\varepsilon\phi_y(z))\cup\partial_\varepsilon f(z)]+\mathrm{N}_{\mathrm{dom}f\cap\mathrm{dom}\Phi}(z)\right),\tag{4.13}$$

where,

$$Y_+^\varepsilon(z):=\{y\in Y|\phi_y(z)\geq-\varepsilon\}\quad\text{for any } z\in X \text{ and } \varepsilon\geq 0.$$

In particular, we write $Y_+(z)$ for $Y_+^0(z)$, that is,

$$Y_+(z):=Y_+^0(z)=\{y\in Y|\phi_y(z)=0\}.$$

The following theorem plays a crucial role in characterizing the weak sharp minima property for convex infinite optimization problem (4.1) in the next subsection. Recall that $\hat f$ is defined by (4.3).

THEOREM 4.5. *Let $z \in \mathrm{dom} f \cap Z$. Then we have the following assertions.*

(i) *The following inclusion holds:*

$$\partial \hat{f}(z) \supseteq \partial f(z) + [0,1]\mathrm{cl}^*\mathrm{co}(\cup_{y\in Y_+(z)}\partial\phi_y(z)) + \mathrm{N}_\Omega(z) \cap \mathbb{B}. \tag{4.14}$$

(ii) *Assume that* $\mathrm{ri}\,(\mathrm{dom} f \cap \mathrm{dom}\Phi) \neq \emptyset$. *Then*

$$\partial \hat{f}(z) = \cap_{\varepsilon>0} E(\varepsilon, z) + \mathrm{N}_\Omega(z) \cap \mathbb{B}. \tag{4.15}$$

(iii) *Assume that either f or Φ is continuous at some point $x_0 \in \mathrm{dom} f \cap \mathrm{dom}\Phi$. Then*

$$\partial \hat{f}(z) = \partial f(z) + [0,1] \cap_{\varepsilon>0} \mathrm{cl}^* \Big(\mathrm{co}(\cup_{y\in Y_+^\varepsilon(z)}\partial_\varepsilon\phi_y(z)) + \mathrm{N}_{\mathrm{dom}\Phi}(z)\Big) + \mathrm{N}_\Omega(z) \cap \mathbb{B} \tag{4.16}$$

if $\mathrm{ri}\,(\mathrm{dom}\Phi) \neq \emptyset$, *and*

$$\partial \hat{f}(z) = \partial f(z) + [0,1]\mathrm{cl}^*\mathrm{co}(\cup_{y\in Y_+(z)}\partial\phi_y(z)) + \mathrm{N}_\Omega(z) \cap \mathbb{B} \tag{4.17}$$

if assumption (B1) *holds and each function $\phi_y(\cdot)$ is continuous at z.*

Proof. (i) Let Φ_+ denote the sup-function of the family $\{[\phi_y]_+ | y \in Y\}$, that is,

$$\Phi_+(x) := \sup_{y\in Y} [\phi_y]_+(x) \quad \text{for each } x \in X.$$

Then $\Phi_+(z) = 0$ (as $z \in Z$), and

$$\{y \in Y : \phi_y(z) = \Phi_+(z)\} = Y_+(z).$$

Note that

$$\partial[\phi_y]_+(z) = \begin{cases} [0,1]\partial\phi_y(z), & \text{if } y \in Y_+(z), \\ \{0\}, & \text{otherwise,} \end{cases}$$

(see [39, Corollary 2.8.11 and Example 2.8.1]). Hence, applying Proposition 4.3(i) (to the family $\{[\phi_y]_+ | y \in Y\}$ in place of $\{\phi_y | y \in Y\}$), we have that

$$[0,1]\mathrm{cl}^*\mathrm{co}(\cup_{y\in Y_+(z)}\partial\phi_y(z)) = \mathrm{cl}^*\mathrm{co}(\cup_{y\in Y_+(z)}\partial[\phi_y]_+(z)) \subseteq \partial\Phi_+(z). \tag{4.18}$$

Note further by (2.5) that

$$\partial \mathrm{d}_\Omega(z) = \mathrm{N}_\Omega(z) \cap \mathbb{B}. \tag{4.19}$$

Since $\partial \hat{f}(z) \supseteq \partial f(z) + \partial\Phi_+(z) + \partial \mathrm{d}_\Omega(z)$, inclusion (4.14) follows from (4.18) and (4.19), and thus assertion (i) is proved.

(ii) By Lemma 2.6(iv), one has that

$$\partial \hat{f}(z) = \partial(f + \Phi_+)(z) + \partial \mathrm{d}_\Omega(z).$$

Therefore, to show (4.15), it suffices to verify that

$$\partial(f + \Phi_+)(z) = \cap_{\varepsilon>0} E(\varepsilon, z). \tag{4.20}$$

To do this, consider the family $\{\phi_y | y \in \tilde{Y}\}$, where $\tilde{Y} := Y \cup \{y_\infty\}$ with $\phi_{y_\infty} := 0$. Then, one checks that

$$\Phi_+(\cdot) = \sup_{y\in\tilde{Y}} \phi_y(\cdot), \quad \mathrm{dom}\Phi_+ = \mathrm{dom}\Phi, \tag{4.21}$$

and

$$\tilde{Y}^\varepsilon(z) = \{y \in \tilde{Y} | \phi_y(z) \geq \Phi_+(z) - \varepsilon\} = Y_+^\varepsilon(z) \cup \{y_\infty\} \quad \text{for each } \varepsilon > 0. \tag{4.22}$$

By assumption, Proposition 4.4 can be applied (to the family $\{\phi_y | y \in \tilde{Y}\}$ in place of $\{\phi_y | y \in Y\}$), and so we have from (4.21) and (4.22) that

$$\partial(f + \Phi_+)(z) = \cap_{\varepsilon>0} \mathrm{cl}^* [\mathrm{co}(\cup_{y\in\tilde{Y}^\varepsilon(z)} (\partial_\varepsilon f(z) + \partial_\varepsilon \phi_y(z))) + \mathrm{N}_{\mathrm{dom}(f+\Phi_+)}(z)].$$

In terms of (4.13), this shows (4.20) as $\partial_\varepsilon \phi_{y_\infty}(z) = 0$, and the proof of assertion (ii) is complete.

(iii) By assumption, it follows from Lemma 2.6(iv) that

$$\partial \hat{f}(z) = \partial f(z) + \partial\Phi_+(z) + \partial \mathrm{d}_\Omega(z). \tag{4.23}$$

Clearly, $\Phi_+(\cdot) = \max\{\Phi(\cdot), 0\}$ and $\Phi_+(z) = 0$. Hence, we have that

$$\partial\Phi_+(z) = [0, 1]\partial\Phi(z) \quad \text{and} \quad Y^\varepsilon(z) = Y_+^\varepsilon(z) \quad \text{for each } \varepsilon \geq 0. \tag{4.24}$$

Thus, thanks to (4.23), (4.24) and (4.19), (4.16) and (4.17) follow immediately from (4.5) and (4.6), respectively. The proof is complete. □

Remark 4.6. *Let Φ be sup-function of the family $\{\phi_y | y \in Y\}$ defined by (4.4). Then some subdifferential rules for the function $f + \Phi$ were established in [10]. Noting that f and ϕ_y involved here are lsc for each $y \in Y$, the assumption in [10, Theorem 4] is satisfied. Thus the corresponding results there regarding the subdifferentials are applicable to establishing the counterparts of Theorem 4.5. To this purpose, let $z \in \mathrm{dom} f \cap Z$ and let $\mathcal{F}(z)$ denote the family of finite-dimensional subspaces containing z. Then, using the similar arguments we did for proving (ii) in Theorem 4.5, one can apply [10, Theorem 4] and [10, Corollary 8], together with Lemma 2.6(iv), to obtain the following subdifferential formulas for $\hat{f}$ at z:*

(i) *The following equality holds:*

$$\partial \hat{f}(z) = \cap_{\varepsilon>0, L\in\mathcal{F}(z)} \mathrm{cl}^* \mathrm{co} \left\{ \cup_{y\in Y_+^\varepsilon(z)} (\partial_\varepsilon \phi_y(z) \cup \{0\}) + \partial(f + \delta_{L\cap\mathrm{dom}\Phi})(z) \right\} + \mathrm{N}_\Omega(z) \cap \mathbb{B}. \tag{4.25}$$

(ii) *Assume that* $\mathrm{ri}\,(\mathrm{dom} f \cap \mathrm{dom}\Phi) \neq \emptyset$ *and* $f|_{\mathrm{aff}\,(\mathrm{dom} f\cap\mathrm{dom}\Phi)}$ *is continuous on* $\mathrm{ri}\,(\mathrm{dom} f \cap \mathrm{dom}\Phi)$. *Then*

$$\partial \hat{f}(z) = \cap_{\varepsilon>0} \mathrm{cl}^* \mathrm{co} \left\{ \cup_{y\in Y_+^\varepsilon(z)} (\partial_\varepsilon \phi_y(z) \cup \{0\}) + \partial(f + \delta_{\mathrm{dom}\Phi})(z) \right\} + \mathrm{N}_\Omega(z) \cap \mathbb{B},$$

and

$$\partial \hat{f}(z) = \cap_{\varepsilon>0} \mathrm{cl}^* \mathrm{co} \left\{ \cup_{y\in Y_+^\varepsilon(z)} (\partial_\varepsilon \phi_y(z) \cup \{0\}) + \partial f(z) + \mathrm{N}_{\mathrm{dom}\Phi}(z) \right\} + \mathrm{N}_\Omega(z) \cap \mathbb{B}$$

if f is continuous at some point of $\mathrm{dom}\Phi$.

4.2. Characterizations for the weak sharp minima. By virtue of Proposition 4.2 and Theorems 4.5(i), the following corollary is a direct consequence of Theorems 3.6, 3.8 and 3.9, in which (i) and (ii) were proved in [42, Theorems 3.2 and 3.1] for the case when X is a Banach space, respectively.

COROLLARY 4.7. *Let $\alpha > 0$ and $x_0 \in S$. Consider the following inclusion:*

$$\alpha\mathbb{B} \cap \mathrm{N}_S(z) \subseteq \partial f(z) + [0,1]\mathrm{cl}^*\mathrm{co}(\cup_{y\in Y_+(z)}\partial\phi_y(z)) + \mathrm{N}_\Omega(z) \cap \mathbb{B}. \tag{4.26}$$

Then we have that

(i) *if* (4.26) *holds for each $z \in S$, then problem* (4.1) *has a global weak sharp minima property in S with modulus α;*

(ii) *if there exists $r > 0$ such that* (4.26) *holds for each $z \in S \cap \mathbb{B}(x_0, r)$, then problem* (4.1) *has a local weak sharp minimum property at x_0 in S with modulus α;*

(iii) *if for each $r > 0$ there exists $\alpha(:= \alpha_r) > 0$ such that* (4.26) *holds for each $z \in S \cap r\mathbb{B}$, then problem* (4.1) *has a boundedly weak sharp minima property in S.*

Similarly, based on Theorem 4.5 and Proposition 4.2, and noting the following equality

$$\mathrm{cl}^*(A + B) = \mathrm{cl}^*A + B$$

whenever $A \subseteq X^*$ is an arbitrary set and $B \subseteq X^*$ is a weak*-compact set, one can apply Theorem 3.6 to conclude directly in the following theorem the equivalent characterizations for the global weak sharp minima property of the convex infinite optimization problem (4.1). Theorem 4.8 not only extends [42, Theorem 3.4] under the weaker assumptions, in which only the equivalence between (i) and (4.35) was shown under the assumptions (A1) and (A2) in Banach spaces (noting in section 1 that (B1) and (B2) are strictly weaker than (A1) and (A2)), but also presents more complete characterizations of the global weak sharp minima property in normed spaces, e.g., (ii)-(v) and (4.36)-(4.38) seem new to the best of our knowledge.

THEOREM 4.8. *Let $\alpha > 0$. Suppose that $\mathrm{ri}(\mathrm{dom}f \cap \mathrm{dom}\Phi) \neq \emptyset$. Then we have the following assertions.*

(I) *The following statements are equivalent:*

(i) *Problem* (4.1) *has a global weak sharp minima property in S with modulus α.*

(ii) *For each $z \in S$,*

$$\alpha\mathbb{B} \cap \mathrm{N}_S(z) \subseteq \cap_{\varepsilon>0}E(\varepsilon, z) + \mathrm{N}_\Omega(z) \cap \mathbb{B}. \tag{4.27}$$

(II) *If S is proximal, then* (i) *and* (ii) *are equivalent to the following statement:*

(iii) *For each $z \in S$,*

$$\alpha\mathbb{B} \cap \mathrm{N}_S(z) \subseteq \cap_{\varepsilon>0}E(\varepsilon, z) + [\mathrm{J}^*\mathrm{N}_S(z)]^\circ + \mathrm{N}_\Omega(z) \cap \mathbb{B}. \tag{4.28}$$

(III) *If X is a Hilbert space, then* (i)-(iii) *are equivalent to the following statements:*

(iv) *For each $z \in S$,*

$$\alpha\mathbb{B} \subseteq \mathrm{cl}\{\cap_{\varepsilon>0}E(\varepsilon, z) + [\mathrm{N}_S(z)]^\circ\} + \mathrm{N}_\Omega(z) \cap \mathbb{B}. \tag{4.29}$$

(v) *For any* $z \in S$ *and* $\hat{\alpha} \in (0, \alpha)$,

$$\hat{\alpha}\mathbb{B} \subseteq \cap_{\varepsilon>0} E(\varepsilon, z) + [\mathrm{N}_S(z)]^\circ + \mathrm{N}_\Omega(z) \cap \mathbb{B}. \tag{4.30}$$

Furthermore, assume that either f *or* Φ *is continuous at some point* $x_0 \in \mathrm{dom} f \cap \mathrm{dom}\Phi$. *Then* (4.27)-(4.30) *above can be replaced by*

$$\alpha\mathbb{B} \cap \mathrm{N}_S(z) \subseteq \partial f(z) + [0,1] \cap_{\varepsilon>0} \mathrm{cl}^* \left(\mathrm{co}(\cup_{y \in Y_+^\varepsilon(z)} \partial_\varepsilon \phi_y(z) + \mathrm{N}_{\mathrm{dom}\Phi}(z)) + \mathrm{N}_\Omega(z) \cap \mathbb{B}\right), \tag{4.31}$$

$$\alpha\mathbb{B} \cap \mathrm{N}_S(z) \subseteq \partial f(z) + [0,1] \cap_{\varepsilon>0} \mathrm{cl}^* \left(\mathrm{co}(\cup_{y \in Y_+^\varepsilon(z)} \partial_\varepsilon \phi_y(z) + \mathrm{N}_{\mathrm{dom}\Phi}(z)) + \mathrm{N}_\Omega(z) \cap \mathbb{B} + [\mathrm{J}^*\mathrm{N}_S(z)]^\circ\right), \tag{4.32}$$

$$\alpha\mathbb{B} \subseteq \mathrm{cl}[\partial f(z) + [0,1] \cap_{\varepsilon>0} \mathrm{cl}(\mathrm{co}(\cup_{y \in Y_+^\varepsilon(z)} \partial_\varepsilon \phi_y(z)) + \mathrm{N}_{\mathrm{dom}\Phi}(z)) + [\mathrm{N}_S(z)]^\circ] + \mathrm{N}_\Omega(z) \cap \mathbb{B}, \tag{4.33}$$

$$\hat{\alpha}\mathbb{B} \cap \mathrm{N}_S(z) \subseteq \partial f(z) + [0,1] \cap_{\varepsilon>0} \mathrm{cl}(\mathrm{co}(\cup_{y \in Y_+^\varepsilon(z)} \partial_\varepsilon \phi_y(z)) + \mathrm{N}_{\mathrm{dom}\Phi}(z)) + [\mathrm{N}_S(z)]^\circ + \mathrm{N}_\Omega(z) \cap \mathbb{B}, \tag{4.34}$$

respectively, if $\mathrm{ri}\,(\mathrm{dom}\Phi) \neq \emptyset$, *and by*

$$\alpha\mathbb{B} \cap \mathrm{N}_S(z) \subseteq \partial f(z) + [0,1]\mathrm{cl}^*\mathrm{co}(\cup_{y \in Y_+(z)} \partial \phi_y(z)) + \mathrm{N}_\Omega(z) \cap \mathbb{B}, \tag{4.35}$$

$$\alpha\mathbb{B} \cap \mathrm{N}_S(z) \subseteq \partial f(z) + [0,1]\mathrm{cl}^*\mathrm{co}(\cup_{y \in Y_+(z)} \partial \phi_y(z)) + \mathrm{N}_\Omega(z) \cap \mathbb{B} + [\mathrm{J}^*\mathrm{N}_S(z)]^\circ, \tag{4.36}$$

$$\alpha\mathbb{B} \subseteq \mathrm{cl}[\partial f(z) + [0,1]\mathrm{clco}(\cup_{y \in Y_+(z)} \partial \phi_y(z)) + [\mathrm{N}_S(z)]^\circ + \mathrm{N}_\Omega(z) \cap \mathbb{B}, \tag{4.37}$$

$$\hat{\alpha}\mathbb{B} \cap \mathrm{N}_S(z) \subseteq \partial f(z) + [0,1]\mathrm{clco}(\cup_{y \in Y_+(z)} \partial \phi_y(z)) + [\mathrm{N}_S(z)]^\circ + \mathrm{N}_\Omega(z) \cap \mathbb{B}, \tag{4.38}$$

respectively, if assumptions (B1) *and* (B2) *hold.*

Analogue to Theorem 4.8, we can further apply Theorems 3.8 and 3.9 to obtain in the following theorems the equivalent characterizations for the local or boundedly weak sharp minima property of the convex infinite optimization problem (4.1), respectively. Most of characterizations provided in Theorems 4.9 and 4.10 seem new in the literature, even in the case when the assumptions (A1) and (A2) hold.

THEOREM 4.9. *Let* $\alpha > 0$ *and* $x_0 \in S$. *Suppose that* $\mathrm{ri}(\mathrm{dom} f \cap \mathrm{dom}\Phi) \neq \emptyset$. *Then we have the following assertions.*

(I) *The following statements are equivalent but* (ii) $\Rightarrow$ (i) *for possible different values of* $\alpha > 0$*:*

(i) *Problem* (4.1) *has a local weak sharp minimum property at* x_0 *in* S *with modulus* α.

(ii) *There exists* $r>0$ *such that* (4.27) *holds for each* $z\in S\cap\mathbb{B}(x_0,r)$.

(II) *If* S *is proximal, then* (i) *and* (ii) *are equivalent to the following statements:*

(iii) *There exists* $r>0$ *such that* (4.28) *holds for each* $z\in S\cap\mathbb{B}(x_0,r)$.

(III) *If* X *is a Hilbert space,* (i)-(iii) *are equivalent to the following statement:*

(iv) *There exists* $r>0$ *such that* (4.29) *holds for each* $z\in S\cap\mathbb{B}(x_0,r)$.

(v) *There exists* $r>0$ *such that* (4.30) *holds for any* $z\in S\cap\mathbb{B}(x_0,r)$ *and* $\hat\alpha\in(0,\alpha)$.

Furthermore, assume that either f *or* Φ *is continuous at some point* $x_0\in\operatorname{dom}f\cap\operatorname{dom}\Phi$. *Then* (4.27)-(4.30) *can be replaced by* (4.31)-(4.34), *respectively, if* $\operatorname{ri}(\operatorname{dom}\Phi)\neq\emptyset$, *and by* (4.35)-(4.38), *respectively, if assumptions* (B1) *and* (B2) *hold.*

THEOREM 4.10. *Suppose that* $\operatorname{ri}(\operatorname{dom}f\cap\operatorname{dom}\Phi)\neq\emptyset$. *Then we have the following assertions.*

(I) *The following statements are equivalent:*

(i) *Problem* (4.1) *has a boundedly weak sharp minima property in* S.

(ii) *For each* $r>0$, *there exists* $\alpha>0$ *such that* (4.27) *holds for each* $z\in S\cap r\mathbb{B}$.

(II) *If* S *is proximal, then* (i) *and* (ii) *are equivalent to the following statement:*

(iii)*For each* $r>0$, *there exists* $\alpha>0$ *such that* (4.28) *holds for each* $z\in S\cap r\mathbb{B}$.

(III) *If* X *is a Hilbert space, then* (i)-(iii) *are equivalent to the following statement:*

(iv) *For each* $r>0$, *there exists* $\alpha>0$ *such that*

$$\alpha\mathbb{B}\subseteq\cap_{\varepsilon>0}E(\varepsilon,z)+[\mathrm{N}_S(z)]^\circ+\mathrm{N}_\Omega(z)\cap\mathbb{B}. \tag{4.39}$$

holds for each $z\in S\cap r\mathbb{B}$.

Furthermore, assume that either f *or* Φ *is continuous at some point* $x_0\in\operatorname{dom}f\cap\operatorname{dom}\Phi$. *Then* (4.27), (4.28) *and* (4.39) *can be replaced by* (4.31), (4.32) *and*

$$\alpha\mathbb{B}\subseteq\partial f(z)+[0,1]\cap_{\varepsilon>0}\operatorname{cl}(\operatorname{co}(\cup_{y\in Y_+^\varepsilon(z)}\partial_\varepsilon\phi_y(z))+\mathrm{N}_{\operatorname{dom}\Phi}(z))+[\mathrm{N}_S(z)]^\circ+\mathrm{N}_\Omega(z)\cap\mathbb{B},$$

respectively, if $\operatorname{ri}(\operatorname{dom}\Phi)\neq\emptyset$, *and by* (4.35), (4.36) *and*

$$\alpha\mathbb{B}\subseteq\partial f(z)+[0,1]\operatorname{clco}(\cup_{y\in Y_+(z)}\partial\phi_y(z))+[\mathrm{N}_S(z)]^\circ+\mathrm{N}_\Omega(z)\cap\mathbb{B},$$

respectively, if assumptions (B1) *and* (B2) *hold.*

REMARK 4.11. *Involving the subdifferential formulas for* $\hat f$ *presented in Remark 4.6 into the considerations of characterizing the global/local/boundedly weak sharp minima property of the convex infinite optimization problem* (4.1), *one can establish similarly the counterparts of Theorems 4.8-4.10, which provide some other characterizations for the corresponding weak sharp minima properties. In particular, the characterization results based on the subdifferential formula* (4.25) *do not require the blanket assumption* $\operatorname{ri}(\operatorname{dom}f\cap\operatorname{dom}\Phi)\neq\emptyset$ *made in Theorems 4.8-4.10.*

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