Modified inexact Levenberg-Marquardt methods for solving nonlinear least squares problems

Ji-Feng Bao* Carisa Kwok Wai Yu† Jinhua Wang‡ Yaohua Hu§
Jen-Chih Yao¶

Abstract. In the present paper, we propose a modified inexact Levenberg-Marquardt method (LMM) and its global version by virtue of Armijo, Wolfe or Goldstein line-search schemes to solve nonlinear least squares problems (NLSP), especially for the underdetermined case. Under a local error bound condition, we show that a sequence generated by the modified inexact LMM converges to a solution superlinearly and even quadratically for some special parameters, which improves the corresponding results of the classical inexact LMM in [Optim. Methods Softw. 17 (2002): pp. 605-626]. Furthermore, the quadratical convergence of the global version of the modified inexact LMM is also established. Finally, preliminary numerical experiments on some medium/large scale underdetermined NLSP show that our proposed algorithm outperforms the classical inexact LMM.

Key words. nonlinear least squares problems, inexact Levenberg-Marquardt method, Lipschitz condition, local error bound.

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*School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan, Zhejiang 316022, P. R. China. Key Laboratory of Oceanographic Big Data Mining & Application of Zhejiang Province, Zhoushan, Zhejiang 316022, P. R. China (feelwinter@163.com). This author's work was supported in part by the National Natural Science Foundation of China (11771398), Zhejiang Provincial Natural Science Foundation of China (LY18A010024) and the Scientific Research Foundation of Zhejiang Ocean University (21065013613).

†Department of Mathematics and Statistics, The Hang Seng University of Hong Kong, Hong Kong (carisayu@hsu.edu.hk). This author's work was supported in part by the Research Grants Council of the Hong Kong Special Administrative Region, China (grant UGC/FDS14/P02/15 and UGC/FDS14/P02/17).

‡Corresponding author. Department of Mathematics, Zhejiang University of Technology, Hangzhou 310032, P. R. China (wjh@zjut.edu.cn). This author's work was supported in part by the National Natural Science Foundation of China (11771397) and by Zhejiang Provincial Natural Science Foundation of China (LY17A010021, LY17A010006).

§Shenzhen Key Laboratory of Advanced Machine Learning and Applications, College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China (mayhhu@szu.edu.cn). This author's work was supported in part by the National Natural Science Foundation of China (11601343, 11871347), Natural Science Foundation of Guangdong (2016A030310038), Natural Science Foundation of Shenzhen (JCYJ20170817100950436) and Interdisciplinary Innovation Team of Shenzhen University.

¶Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung, Taiwan (yaojc@maill.cmu.edu.tw). This author's work was supported in part by the National Science Council of Taiwan (MOST 105-2115-M-039-002-MY3).

1 Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Fréchet differentiable function with its continuous Fréchet derivative denoted by f'. Consider the following nonlinear least squares problem (NLSP)

$$\min_{x \in \mathbb{R}^n} \phi(x) := \frac{1}{2}\|f(x)\|^2, \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean norm. Applications for this type of problem can be found in chemistry, physics, finance, economics and so on; see [7, 11, 31, 44] and references therein.

Newton's method is one of the most important algorithms for solving NLSP (1.1) (see [5, 13–15, 53] and references therein), which in generally converges quadratically. However, it requires the computation of the Hessian matrix of ϕ at each iteration

$$\nabla^2\phi(x_k) := f'(x_k)^T f'(x_k) + G(x_k),$$

where $G(x_k)$ contains the second-order Fréchet derivative of f, which may cost expensive, especially for large scale problems. In order to make the procedure more efficient, Gauss-Newton (GN) method (see [24, 35, 39]) was proposed to obtain the search direction d_k by solving the following minimization problem

$$\min_{d \in \mathbb{R}^n} \|f(x_k) + f'(x_k)d\|^2,$$

that is,

$$d_k := -f'(x_k)^\dagger f(x_k),$$

where $f'(x_k)^\dagger$ is the Moore-Penrose inverse of $f'(x_k)$. Obviously, d_k is also the solution of the following linear equation (see [7, 23])

$$f'(x_k)^T f'(x_k)d = -f'(x_k)^T f(x_k). \tag{1.2}$$

Note that in the case when $f'(x_k)^T f'(x_k)$ is singular, the cost of the GN method is expensive. To avoid this disadvantage, the Levenberg-Marquardt method (LMM) (see, e.g., [32, 40, 41]) was introduced, where the direction d_k is provided by solving the following symmetric positive definite linear equation

$$\left(f'(x_k)^T f'(x_k) + \lambda_k \mathbb{I}_n\right) d = -f'(x_k)^T f(x_k), \tag{1.3}$$

where $\mathbb{I}_n$ is the identity matrix in $\mathbb{R}^{n\times n}$ and $\lambda_k > 0$ is a given parameter. Clearly, solving (1.3) is equivalent to solving the following regularized optimization problem

$$\min_{d \in \mathbb{R}^n} \psi_k(d) := \|f(x_k) + f'(x_k)d\|^2 + \lambda_k\|d\|^2. \tag{1.4}$$

Setting $\lambda_k := \|f(x_k)\|^\delta$ with $\delta \in [1, 2]$, Fan and Yuan [19] showed that a sequence $\{x_n\}$ generated by the LMM converges quadratically to a solution under a local error bound condition, which is weaker than the condition that $f'(x^*)$ is nonsingular (noting that these results extend the corresponding ones in [51], in which only the quadratical convergence of $\{\mathrm{d}(x_n, S)\}$ was established, where $S := \{x|\ f(x) = 0\}$). Other works about different choices of the parameter λ_k can be found in [16, 20] and references therein.

Note that, the inexact versions of numerical algorithms are much more attractive in practical applications because exactly solving the subproblems (1.3) is very expensive, especially for large scale problems. Hence, Dan et al. [12] introduced an inexact Levenberg-Marquardt method (LMM) for solving NLSP (1.1), in which subproblem (1.3) is solved approximately at each iteration. The inexact LMM is formally presented as follows.

Algorithm ILM

Step 0 Choose an initial point $x_0 \in \mathbb{R}^n$ and a parameter $\delta \in (0, 2]$, and set $k := 0$.

Step 1 If x_k satisfies a stopping criterion, then stop.

Step 2 Set $\lambda_k := \eta\|f(x_k)\|^\delta$. Take a residual control $\epsilon_k > 0$ and calculate d_k by solving approximately the subproblem (1.3) such that the residual

$$r_k := \left(f'(x_k)^T f'(x_k) + \lambda_k \mathbb{I}_n\right) d_k + f'(x_k)^T f(x_k)$$

satisfies

$$\|r_k\| \leq \epsilon_k. \tag{1.5}$$

Step 3 Set $x_{k+1} := x_k + d_k$ and $k := k + 1$. Go to Step 1.

The convergence properties of Algorithm ILM, under the local error bound assumption, were studied in [12, 17, 18], where main interests were focused on stopping criterion (1.5) for the subproblem in Step 2. In particular, in [12, Theorems 2.1 and 2.2], the convergence of $\{x_k\}$ and the superlinear (resp. quadratical) convergence of $\{\mathrm{d}(x_k, S)\}$ were established for the residual control $\epsilon_k := \lambda_k o(\|f(x_k)\|)$ (resp. $\epsilon_k := \lambda_k O(\|f(x_k)\|^2)$, which was weaken to $\epsilon_k := \sqrt{\lambda_k} O(\|f(x_k)\|^2)$ in [17, Theorem 2.2]); while, in [18], the convergence rates were discussed for the relaxed residual control: $\epsilon_k \leq O(\|f(x_k)\|^{\alpha+\xi})$, where $\alpha \in (0, 4)$, $\xi > 0$ and, as a consequence, Algorithm ILM converges quadratically if $\xi \geq 1$ and $\alpha \in [1, 2]$ (see [18, Theorem2.3]).

In the present paper, we particularly focus on the NLSP (1.1) in the underdetermined case (i.e., $m \ll n$), which is found to be applicable in various areas; see [3, 12, 23, 39, 46] and references therein. As the underlying problem size of (1.2) will be large in this case, solving subproblems (1.3) may be expensive for large scale problems. In order to overcome this disadvantage, under the full row rank assumption of the Jacobian $f'(x_k)$, Bao et al. [3] proposed the procedure to obtain d_k, in which one approaches s_k by solving

$$f'(x_k) f'(x_k)^T s = -f(x_k) \tag{1.6}$$

and then determines $d_k := f'(x_k)^T s_k$. The numerical experiments in [3] revealed that this switch scheme is more effective than the one based on (1.2). However, if the Jacobian is not of full row rank, then the linear system (1.6) is inconsistent and so s_k will not be well defined. To fill this gap, inspired by the idea of LMM, we propose in the present paper a modified exact LMM by combining the regularization technique and switch scheme (1.6). That is, for each iteration, given a regularization parameter $\lambda_k > 0$, the search direction d_k is calculated by solving s_k via

$$\left(f'(x_k)f'(x_k)^T + \lambda_k \mathbb{I}_m\right) s = -f(x_k) \tag{1.7}$$

and setting $d_k := f'(x_k)^T s_k$. This modified exact LMM seems new in the literature to the best of our knowledge. The advantage of the proposed modified LMM over the ones in [12, 17, 18] is significant in the underdetermined case, in which the underlying problem size of (1.7) is much less than that of (1.3), and hence solving subproblems (1.7) is less expensive than (1.3). Inspired by [12], in the present paper, we introduce a modified inexact LMM by approximately solving problem (1.7), and investigate its convergence analysis, where the residual satisfies $\|r_k\| \le \theta_k \|f(x_k)\|^\nu$ with $\{\theta_k\}$ being a bounded positive sequence, $\nu \in \left[1 + \frac{\delta}{2}, 2\right]$ and $\delta \in (0, 1]$. In particular, the modified inexact LMM converges to a solution of (1.1) superlinearly and even quadratically for some specific parameters under a local Hölder error bound condition, which is a more general assumption than the classical error bound condition assumed in [12, 17, 18]. It is worth mentioning that the Hölder error bound condition has also been used to explore the convergence rate of the exact LMM (based on the standard subproblem (1.3)) with the different choices of the LM parameters $\{\lambda_k\}$ in (1.3): $\lambda_k := \eta \|f(x_k)\|^\sigma$ with $\sigma \in [1, 2]$ and $\sigma \in (0, 4 - \sqrt{2}]$ in [22] and [54], respectively, where $\eta > 0$, and an adaptive LM parameter in [1] (see [1, Algorithm LLM in page 7] for more details).

Note that the modified inexact LMM is guaranteed to converge fast when starting from an initial point near a solution of NLSP (1.1). However, the selection of a high quality initial point is a difficult issue, which may hinder the implementation of the modified inexact LMM (as well as the classical LMM) in practical applications. To avoid this difficulty, we further propose a globalization strategy for the modified inexact LMM by virtue of the Armijo, Wolfe or Goldstein line-search schemes, and show that the generated sequence converges to a solution quadratically under the local Hölder error bound condition. Preliminary numerical results illustrate that the global version of the modified inexact LMM is more efficient than the one proposed in [12] for solving some underdetermined NLSP.

The remainder of the paper is organized as follows. In section 2, we present some notions and preliminary results. We propose a modified inexact LMM and establish its local convergence rates under a local error bound condition of Hölder order in section 3. In section 4, a globalized version of the modified inexact LMM with Armijo, Wolfe or Goldstein line-search strategy is presented and its global convergence theorem is provided. Numerical experiments for the underdetermined NLSP are reported in section 5.

2 Notations and preliminary results

We consider the n-dimensional Euclidean space $\mathbb{R}^n$ with Euclidean norm $\|\cdot\|$. For $x \in \mathbb{R}^n$ and $r > 0$, we use $\mathbf{B}(x, r)$ to denote the closed ball with radius r and center x. For $W \subseteq \mathbb{R}^n$, we use $\mathrm{d}(x, W)$ and $P_W(x)$ to denote the Euclidean distance of x from W and the projection of x onto W, respectively, that is,

$$\mathrm{d}(x, W) = \inf\{\|x - y\| | \ y \in W\} \quad \text{and} \quad P_W(x) = \{y \in W | \ \|x - y\| = \mathrm{d}(x, W)\}.$$

Let $\mathbb{R}^{m\times n}$ be the space of all $m \times n$ real matrices and $\mathbb{I}_n$ be the identity matrix in $\mathbb{R}^{n\times n}$. Let $A \in \mathbb{R}^{m\times n}$ and use A^T to denote the transpose of A. The matrix $A^\dagger \in \mathbb{R}^{n\times m}$ is said to be the Moore-Penrose inverse of A if it satisfies the following four equalities:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^T = AA^\dagger, \quad (A^\dagger A)^T = A^\dagger A.$$

In particular, if $\mathrm{rank}(A) = l$ and the singular value decomposition (SVD) of A is

$$A = U\Sigma V^T, \tag{2.1}$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and

$$\Sigma = \begin{pmatrix} \Sigma_l & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \Sigma_l = \mathrm{diag}\{\sigma_1, \cdots, \sigma_l\} \text{ with singular values } \sigma_1 \geq \cdots \geq \sigma_l > 0. \tag{2.2}$$

Then one has

$$A^\dagger = V\Sigma^\dagger U^T.$$

The following lemmas describe some useful properties about matrix computation, in which Lemma 2.1 is taken from [47, Theorem 4.11].

Lemma 2.1. *Let $A, B \in \mathbb{R}^{m\times n}$ be matrices with singular values given by*

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \quad \text{and} \quad \tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \cdots \geq \tilde{\sigma}_p,$$

respectively. Then

$$\|\mathrm{diag}(\tilde{\sigma}_1 - \sigma_1, \cdots, \tilde{\sigma}_p - \sigma_p, 0, \cdots, 0)\| \leq \|B - A\|.$$

Lemma 2.2. *Let $A \in \mathbb{R}^{m\times n}$ be a matrix, and let $\lambda > 0$ be a constant. Then*

$$\left(A^T A + \lambda\mathbb{I}_n\right)^{-1} A^T = A^T \left(AA^T + \lambda\mathbb{I}_m\right)^{-1}, \tag{2.3}$$

$$\|A^T \left(AA^T + \lambda\mathbb{I}_m\right)^{-1}\| \leq \frac{1}{2\sqrt{\lambda}}, \tag{2.4}$$

$$\|AA^T \left(AA^T + \lambda\mathbb{I}_m\right)^{-1}\| \leq 1, \tag{2.5}$$

and

$$\|\left(A^T A + \lambda\mathbb{I}_n\right)^{-1}\| \leq \frac{1}{\lambda}. \tag{2.6}$$

Proof. It is obvious that

$$A^T\left(AA^T+\lambda\mathbb{I}_m\right)=\left(A^TA+\lambda\mathbb{I}_n\right)A^T,$$

and so (2.3) is seen to hold.

Suppose that $\operatorname{rank}(A)=l\le\min\{m,n\}$. Then, by the SVD of A, there exist orthogonal matrices $U\in\mathbb{R}^{m\times m}$ and $V\in\mathbb{R}^{n\times n}$ such that (2.1) and (2.2) are satisfied. Hence, it follows that

$$A^T\left(AA^T+\lambda\mathbb{I}_m\right)^{-1}=V\Sigma^T\left(\Sigma\Sigma^T+\lambda\mathbb{I}_m\right)^{-1}U^T \tag{2.7}$$

and

$$\left(A^TA+\lambda\mathbb{I}_n\right)^{-1}=V\left(\Sigma^T\Sigma+\lambda\mathbb{I}_n\right)^{-1}V^T. \tag{2.8}$$

Then, it follows from (2.1), (2.7) and (2.8) that

$$\|A^T\left(AA^T+\lambda\mathbb{I}_m\right)^{-1}\|=\|\Sigma^T\left(\Sigma\Sigma^T+\lambda\mathbb{I}_m\right)^{-1}\|=\max_{1\le i\le l}\left\{\frac{\sigma_i}{\sigma_i^2+\lambda}\right\}\le\frac{1}{2\sqrt{\lambda}},$$

$$\|AA^T\left(AA^T+\lambda\mathbb{I}_m\right)^{-1}\|=\|\Sigma\Sigma^T\left(\Sigma\Sigma^T+\lambda\mathbb{I}_m\right)^{-1}\|=\max_{1\le i\le l}\left\{\frac{\sigma_i^2}{\sigma_i^2+\lambda}\right\}\le 1,$$

and

$$\|\left(A^TA+\lambda\mathbb{I}_n\right)^{-1}\|=\|\left(\Sigma^T\Sigma+\lambda\mathbb{I}_n\right)^{-1}\|=\max_{1\le i\le l}\left\{\frac{1}{\sigma_i^2+\lambda}\right\}\le\frac{1}{\lambda}.$$

Therefore, (2.4), (2.5) and (2.6) are seen to hold. The proof is completed. □

Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a Fréchet differentiable function with its continuous Fréchet derivative denoted by f'. Recall that f' is said to be Lipschitz continuous on $D\subset\mathbb{R}^n$ with modulus $L>0$ if

$$\|f'(y)-f'(x)\|\le L\|y-x\|,\qquad\forall x,y\in D; \tag{2.9}$$

and f' is said to be local Lipschitz continuous around $\hat{x}\in\mathbb{R}^n$ if there exist $L>0$ and $r>0$ such that (2.9) holds with $\mathbf{B}(\hat{x},r)$ in place of D.

Remark 2.1. *If f' is Lipschitz continuous on $\mathbf{B}(\hat{x},r)$ with modulus L, then it is easy to see that*

$$\|f(y)-f(x)-f'(x)(y-x)\|\le\frac{L}{2}\|y-x\|^2,\qquad\forall x,y\in\mathbf{B}(\hat{x},r), \tag{2.10}$$

and there exists a constant $L_1>0$ such that

$$\|f(y)-f(x)\|\le L_1\|y-x\|,\qquad\forall x,y\in\mathbf{B}(\hat{x},r). \tag{2.11}$$

The notion of error bounds, introduced by Hoffman [25] for convex inequality systems, plays an important role in the treatment of various issues in mathematical programming and has been extensively studied by many researchers; see [43, 45, 52] and the references therein. As a natural extension, error bounds of Hölder order have been introduced for polynomial systems in [37] and widely explored in [33, 34, 38] and references therein. Here we consider a nonlinear equation

$$f(x) = 0, \tag{2.12}$$

and denote the set of its roots by

$$S := \{x |\, f(x) = 0\}.$$

The notion of the error bound of Hölder order is recalled as follows.

Definition 2.1. *Let $\mu > 0$, $\beta > 0$, $x^* \in S$ and $D \subseteq \mathbb{R}^n$. Equation* (2.12) *is said to have*

(i) *an error bound of order β on D with modulus μ if*

$$\mu \mathrm{d}(x, S) \le \|f(x)\|^\beta, \qquad \forall x \in D; \tag{2.13}$$

(ii) *a local error bound of order β around x^* if there exist $\hat{\mu} > 0$ and $r > 0$ such that equation* (2.12) *has an error bound of order β on $\mathbf{B}(x^*, r)$ with modulus $\hat{\mu}$.*

In particular, equation (2.12) *is said to have an error bound on D with modulus μ if* (2.13) *holds for $\beta = 1$.*

The notions of error bounds have been widely applied in the sensitivity and convergence analysis of many optimization algorithms. In particular, the error bound or/and local error bound (with $\beta = 1$) is applied in the convergence study of the LMM or inexact LMM; see [4, 12, 16, 19–21, 51].

Remark 2.2. *The notion of weak sharp minima has been extensively studied and widely used to analyze the convergence properties of many optimization algorithms; see [9, 10, 28, 36, 50] and references therein. As a natural extension, the notion of weak sharp minima of order β ($\beta > 1$) has been investigated and applied in [8, 27, 29, 45, 48]. It is well-known that equation* (2.12) *has an error bound of order β on D if and only if S is the set of weak sharp minima of order $\frac{1}{\beta}$ for optimization problem* (1.1) *on D with modulus $\mu^{\frac{1}{\beta}}$ (see, e.g., [28, 48]).*

3 A modified inexact LMM and its local convergence analysis

This section aims to propose a modified inexact Levenberg-Marquardt method (LMM) to solve the NLSP (1.1) and establish its local convergence under the local error bound condition. The modified inexact LMM is formally stated as follows.

Algorithm 3.1

Step 0 Choose an initial point $x_0 \in \mathbb{R}^n$, parameters $\delta \in (0,1]$ and $\nu \in \left[1+\frac{\delta}{2}, 2\right]$, and a sequence $\{\theta_k\} \subseteq \mathbb{R}_+$ such that $\sup_{k\in\mathbb{N}} \theta_k < 1$. Set $k := 0$.

Step 1 If $\|f(x_k)\| = 0$, then stop.

Step 2 Set $\lambda_k := \|f(x_k)\|^\delta$, and calculate s_k by solving approximately the following subproblem

$$\left(f'(x_k)f'(x_k)^T + \lambda_k \mathbb{I}_m\right) s = -f(x_k) \tag{3.1}$$

such that the residual

$$r_k := \left(f'(x_k)f'(x_k)^T + \lambda_k \mathbb{I}_m\right) s_k + f(x_k) \tag{3.2}$$

satisfies

$$\|r_k\| \le \theta_k \|f(x_k)\|^\nu, \tag{3.3}$$

and set $d_k := f'(x_k)^T s_k$.

Step 3 Set $x_{k+1} := x_k + d_k$ and $k := k+1$. Go to Step 1.

Note that the main difference between Algorithm 3.1 and Algorithm ILM is that (1.3) in Algorithm ILM is replaced by (3.1) in Algorithm 3.1. In the underdetermined case, the underlying problem size of (3.1) is much less than that of (1.3), and so, solving subproblems (3.1) is less expensive than (1.3) for large scale problems.

Remark 3.1. *The sequence generated by Algorithm 3.1 is well defined. Indeed, if x_k is not a solution of the equation $f(x) = 0$, one sees that $\lambda_k = \|f(x_k)\|^\delta$ is always positive; hence, the linear system* (3.1) *is symmetric and positive definite, and it has a unique solution. Thus, the approximate solution s_k satisfying* (3.3) *always exists, and so d_k and x_{k+1} are well generated.*

Remark 3.2. *Algorithm 3.1 is well defined and all the main theorems in the present paper remain true for the more general parameters setting such as $\delta \in (0,2]$. Here, we restrict the parameters $\delta \in (0,1]$ and $\nu \in [1+\frac{\delta}{2}, 2]$ for the consideration of computation efficiency. Indeed, in the case when $f(x_k)$ is near* 0*, if δ becomes smaller, then $\lambda_k = \|f(x_k)\|^\delta$ becomes larger and so the condition number of the coefficient matrix $f'(x_k)f'(x_k)^T + \lambda_k \mathbb{I}_m$ for solving the linear equations in* (3.1) *will be smaller; on the other hand, if ν becomes smaller, then $\epsilon_k = \theta_k \|f(x_k)\|^\nu$ becomes larger and so less computation is required in* (3.3)*. For this consideration, small values of parameters δ and ν will be preferred as x_k will eventually close to the solution of* (2.12)*. We remark that Theorem 3.1, as well as the arguments presented for its proof below, remain true if assumption $\delta \in (0,2]$ is in place of $\delta \in (0,1]$.*

Below, we establish a main theorem about convergence results for a sequence generated by Algorithm 3.1. Our proof follows a line of analysis similar to that of [17, 18] with modifications to deal with different subproblem (1.7) and different error estimates under a more general error bound condition. We begin with several lemmas, which are beneficial to the proof of the main theorem. For this purpose, let $\{x_k\}$ be a sequence generated by Algorithm 3.1 (together with the associated sequence $\{d_k\}$). Set

$$d_k^* := -f'(x_k)^T \left(f'(x_k)f'(x_k)^T + \lambda_k \mathbb{I}_m\right)^{-1} f(x_k) \quad \text{for each } k \geq 0. \tag{3.4}$$

Then, by (2.3) (with $f'(x_k)$ in place of A), we have

$$d_k^* = -\left(f'(x_k)^T f'(x_k) + \lambda_k \mathbb{I}_n\right)^{-1} f'(x_k)^T f(x_k). \tag{3.5}$$

By the optimality condition, d_k^* is the solution of (1.4). Noting further, by (3.2), we have the relationship between d_k and d_k^* as follows:

$$d_k = f'(x_k)^T s_k = d_k^* + f'(x_k)^T \left(f'(x_k)f'(x_k)^T + \lambda_k \mathbb{I}_m\right)^{-1} r_k. \tag{3.6}$$

This implies that

$$\begin{aligned} \|d_k\| &\leq \|d_k^*\| + \|f'(x_k)^T \left(f'(x_k)f'(x_k)^T + \lambda_k \mathbb{I}_m\right)^{-1}\| \cdot \|r_k\| \\ &\leq \|d_k^*\| + \frac{\|r_k\|}{2\sqrt{\lambda_k}}, \end{aligned} \tag{3.7}$$

where the second inequality follows from (2.4) with $A = f'(x_k)$. Let $\psi_k : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\psi_k(d) := \|f(x_k) + f'(x_k)d\|^2 + \lambda_k \|d\|^2 \quad \text{ for each } d \in \mathbb{R}^n.$$

Lemma 3.1. *Let $k \in \mathbb{N}$ and let d_k^* be giving by (3.4). Then, we have*

$$\|f(x_k) + f'(x_k)d_k\| \leq \|f(x_k) + f'(x_k)d_k^*\| + \|r_k\| \tag{3.8}$$

and

$$\psi_k(d_k^*) \leq \psi_k(d) \quad \textit{ for each } d \in \mathbb{R}^n. \tag{3.9}$$

Proof. Note that

$$\begin{aligned} &\|f(x_k) + f'(x_k)d_k\| \\ &= \|f(x_k) + f'(x_k)d_k^* + f'(x_k)f'(x_k)^T \left(f'(x_k)f'(x_k)^T + \lambda_k \mathbb{I}_m\right)^{-1} r_k\| \\ &\leq \|f(x_k) + f'(x_k)d_k^*\| + \|f'(x_k)f'(x_k)^T \left(f'(x_k)f'(x_k)^T + \lambda_k \mathbb{I}_m\right)^{-1}\|\|r_k\| \\ &\leq \|f(x_k) + f'(x_k)d_k^*\| + \|r_k\|, \end{aligned}$$

where the last inequality follows from (2.5) with $A = f'(x_k)$. Hence, (3.8) is seen to hold. Note that d_k^* is the solution of (1.4) and so (3.9) follows. □

Recall that S and d_k are defined by (2.12) and (3.6), respectively, and we write

$$\theta := \sup_{k \in \mathbb{N}} \theta_k < 1. \tag{3.10}$$

Below, we present some crucial properties related to d_k.

Lemma 3.2. *Let $0 < r_0 < 1$ and $x^* \in S$. Suppose that*

(a) *f' is Lipschitz continuous on $\mathbf{B}(x^*, r_0)$ with modulus $L > 0$;*

(b) *(2.12) has an error bound of order β on $\mathbf{B}(x^*, r_0)$ with modulus $\mu \geq 0$.*

Suppose further that $0 < \delta \leq 2\beta$. Then, there exists $c > 0$ such that, if $x_k, x_{k+1} \in \mathbf{B}(x^*, \frac{r_0}{2})$, the following inequalities hold:

$$\|d_k\| \leq c\mathrm{d}(x_k, S), \tag{3.11}$$

$$\|f(x_k) + f'(x_k)d_k\| \leq c\mathrm{d}(x_k, S)^{1+\frac{\delta}{2}} \tag{3.12}$$

and

$$\mathrm{d}(x_{k+1}, S) \leq c\mathrm{d}(x_k, S)^{\beta(1+\frac{\delta}{2})}. \tag{3.13}$$

Proof. (i) Let

$$c_1 := \frac{1}{2}\sqrt{L^2\mu^{-\frac{\delta}{\beta}} + 4} + \frac{\theta}{2}L_1^{\nu-\frac{\delta}{2}}, c_2 := \frac{1}{2}\sqrt{L^2 + 4L_1^{\delta}} + \theta L_1^{\nu}, c_3 := \frac{(2c_2 + Lc_1^2)^{\beta}}{\mu 2^{\beta}}$$

and

$$c := \max\{c_1, c_2, c_3\}.$$

Note by (3.7) that

$$\|d_k\| \leq \|d_k^*\| + \frac{\|r_k\|}{2\sqrt{\lambda_k}}. \tag{3.14}$$

Let $\bar{x}_k \in P_S(x_k)$. Since $x^* \in S$ and $x_k \in \mathbf{B}(x^*, \frac{r_0}{2})$, by the definition of $\bar{x}_k$, we have

$$\|\bar{x}_k - x_k\| \leq \|x_k - x^*\| \leq \frac{r_0}{2} < 1 \tag{3.15}$$

and

$$\|\bar{x}_k - x^*\| \leq \|\bar{x}_k - x_k\| + \|x_k - x^*\| \leq 2\|x_k - x^*\| \leq r_0,$$

which implies that $\bar{x}_k \in \mathbf{B}(x^*, r_0)$. Then, as $\lambda_k = \|f(x_k)\|^{\delta}$ and $\nu \geq 1 + \frac{\delta}{2}$, it follows from (3.3), (3.10), (2.11) and (3.15) that

$$\frac{\|r_k\|}{2\sqrt{\lambda_k}} \leq \frac{\theta\|f(x_k)\|^{\nu}}{2\|f(x_k)\|^{\frac{\delta}{2}}} = \frac{\theta}{2}\|f(x_k) - f(\bar{x}_k)\|^{\nu-\frac{\delta}{2}} \leq \frac{\theta}{2}L_1^{\nu-\frac{\delta}{2}}\|\bar{x}_k - x_k\| = \frac{\theta}{2}L_1^{\nu-\frac{\delta}{2}}\mathrm{d}(x_k, S). \tag{3.16}$$

Below, we estimate the value of $\|d_k^*\|$. Combining (3.9) with the definition of ψ_k yields that

$$\begin{aligned}\|d_k^*\|^2 &\leq \psi_k(d_k^*)/\lambda_k \\ &\leq \psi_k(\bar{x}_k - x_k)/\lambda_k \\ &= \|f(x_k) + f'(x_k)(\bar{x}_k - x_k)\|^2/\lambda_k + \|\bar{x}_k - x_k\|^2.\end{aligned} \tag{3.17}$$

Note by (2.13), (2.11) and the definition of $\bar{x}_k$ that

$$L_1^{\delta}\|\bar{x}_k - x_k\|^{\delta} \geq \|f(x_k) - f(\bar{x}_k)\|^{\delta} = \lambda_k = \|f(x_k)\|^{\delta} \geq \mu^{\frac{\delta}{\beta}}\|\bar{x}_k - x_k\|^{\frac{\delta}{\beta}}. \tag{3.18}$$

Observe further from (2.10) that

$$\|f(x_k)+f'(x_k)(\bar{x}_k-x_k)\| = \|f(\bar{x}_k)-f(x_k)-f'(x_k)(\bar{x}_k-x_k)\| \le \frac{L}{2}\|\bar{x}_k-x_k\|^2. \quad (3.19)$$

This, together with (3.18), implies that

$$\|f(x_k)+f'(x_k)(\bar{x}_k-x_k)\|^2/\lambda_k \le \frac{L^2}{4}\mu^{-\frac{\delta}{\beta}}\|\bar{x}_k-x_k\|^{4-\frac{\delta}{\beta}} \le \frac{L^2}{4}\mu^{-\frac{\delta}{\beta}}\left(\frac{r_0}{2}\right)^{2-\frac{\delta}{\beta}}\|\bar{x}_k-x_k\|^2, \quad (3.20)$$

where the last inequality holds because of (3.15) and $\beta \ge \frac{\delta}{2}$. Then, as $0 < r_0 < 1$, it follows from (3.17) and (3.20) that

$$\|d_k^*\| \le \sqrt{\frac{L^2}{4}\mu^{-\frac{\delta}{\beta}}+1}\cdot\|\bar{x}_k-x_k\| = \frac{1}{2}\sqrt{L^2\mu^{-\frac{\delta}{\beta}}+4}\cdot \mathrm{d}(x_k,S),$$

Combining this with (3.14) and (3.16) gives that

$$\|d_k\| \le \left(\frac{1}{2}\sqrt{L^2\mu^{-\frac{\delta}{\beta}}+4}+\frac{\theta}{2}L_1^{\nu-\frac{\delta}{2}}\right)\mathrm{d}(x_k,S) = c_1\mathrm{d}(x_k,S) \le c\mathrm{d}(x_k,S). \quad (3.21)$$

Hence, (3.11) is seen to hold.

Below, we show that (3.12) holds. In fact, by (3.8), we have

$$\|f(x_k)+f'(x_k)d_k\| \le \|f(x_k)+f'(x_k)d_k^*\|+\|r_k\|. \quad (3.22)$$

By (3.9) and the definition of ψ_k, we have

$$\begin{aligned}\|f(x_k)+f'(x_k)d_k^*\|^2 &\le \psi_k(d_k^*)\\ &\le \psi_k(\bar{x}_k-x_k)\\ &= \|f(x_k)+f'(x_k)(\bar{x}_k-x_k)\|^2+\lambda_k\|\bar{x}_k-x_k\|^2. \quad (3.23)\end{aligned}$$

Note by (3.18) and (3.19) that

$$\begin{aligned}\|f(x_k)+f'(x_k)(\bar{x}_k-x_k)\|^2+\lambda_k\|\bar{x}_k-x_k\|^2 &\le \frac{L^2}{4}\|\bar{x}_k-x_k\|^4+L_1^\delta\|\bar{x}_k-x_k\|^{2+\delta}\\ &\le \frac{1}{4}(L^2+4L_1^\delta)\|\bar{x}_k-x_k\|^{2+\delta},\end{aligned}$$

where the last inequality holds by the estimation (3.15). Combining this with (3.23) gives that

$$\|f(x_k)+f'(x_k)d_k^*\| \le \frac{1}{2}\sqrt{L^2+4L_1^\delta}\cdot\|\bar{x}_k-x_k\|^{1+\frac{\delta}{2}}. \quad (3.24)$$

On the other hand, by (3.10), (3.3), and (2.11), one has

$$\|r_k\| \le \theta\|f(x_k)\|^\nu = \theta\|f(x_k)-f(\bar{x}_k)\|^\nu \le \theta L_1^\nu\|\bar{x}_k-x_k\|^\nu.$$

This, together with (3.24), (3.22), (3.15) and $\nu \geq 1 + \frac{\delta}{2}$, gives that

$$\|f(x_k) + f'(x_k)d_k\| \leq \left(\frac{1}{2}\sqrt{L^2 + 4L_1^\delta} + \theta L_1^\nu\right) \|\bar{x}_k - x_k\|^{1+\frac{\delta}{2}} = c_2 \mathrm{d}(x_k, S)^{1+\frac{\delta}{2}}. \tag{3.25}$$

Hence, (3.12) is seen to hold by the definition of c.

Since $x_k, x_{k+1} \in \mathbf{B}(x^*, \frac{r_0}{2})$, it follows from (2.13) and (2.10) that

$$\begin{aligned}
\mathrm{d}(x_{k+1}, S) &\leq \frac{1}{\mu}\|f(x_k + d_k)\|^\beta \\
&\leq \frac{1}{\mu}\left(\|f(x_k + d_k) - f(x_k) - f'(x_k)d_k\| + \|f(x_k) + f'(x_k)d_k\|\right)^\beta \\
&\leq \frac{1}{\mu}\left(\frac{L}{2}\|d_k\|^2 + \|f(x_k) + f'(x_k)d_k\|\right)^\beta.
\end{aligned} \tag{3.26}$$

This, together with (3.21) and (3.25), implies that

$$\mathrm{d}(x_{k+1}, S) \leq \frac{1}{\mu}\left(\frac{Lc_1^2}{2}\mathrm{d}(x_k, S)^2 + c_2 \mathrm{d}(x_k, S)^{1+\frac{\delta}{2}}\right)^\beta \leq c_3 \mathrm{d}(x_k, S)^{\beta(1+\frac{\delta}{2})},$$

where the second inequality follows from the fact that $\delta \in (0, 1]$ and $\mathrm{d}(x_k, S) \leq \|x_k - x^*\| \leq \frac{r_0}{2} < 1$. Hence, (3.13) is seen to hold by the definition of c. The proof is completed. □

Lemma 3.3. *Let $0 < r_0 < 1$ and $x^* \in S$. Suppose that f' is Lipschitz continuous on $\mathbf{B}(x^*, r_0)$ with modulus $L > 0$. If $\{x_k\}$ converges to a solution $\bar{x}$ of (2.12) and $\bar{x} \in \mathbf{B}(x^*, \frac{r_0}{2})$, then there exist a positive constant ω and a positive integer K such that for all $k \geq K$,*

$$\|f(x_k) + f'(x_k)d_k\| \leq \omega \|x_k - \bar{x}\|^{\min\{\nu, 1+\delta\}}. \tag{3.27}$$

Proof. Suppose that $\mathrm{rank}(f'(\bar{x})) = r$ and the singular value decomposition (SVD) of $f'(\bar{x})$ is

$$f'(\bar{x}) = \bar{U}\begin{pmatrix} \bar{\Sigma}_1 & \\ & 0 \end{pmatrix}\bar{V}^T$$

where $\bar{\Sigma}_1 = \mathrm{diag}\{\bar{\sigma}_1, \cdots, \bar{\sigma}_r\}$ with $\bar{\sigma}_1 \geq \cdots \geq \bar{\sigma}_r > 0$, and $\bar{U} \in \mathbb{R}^{m\times m}$ and $\bar{V} \in \mathbb{R}^{n\times n}$ are orthogonal matrices. Since $\{x_k\}$ converges to $\bar{x}$, there exists a positive integer K such that for all $k \geq K$,

$$L\|x_k - \bar{x}\| \leq \min\left\{\frac{\bar{\sigma}_r}{2}, \frac{Lr_0}{2}\right\}. \tag{3.28}$$

Fix $k \geq K$. Suppose the SVD of $f'(x_k)$ is as follows:

$$\begin{aligned}
f'(x_k) = U\Sigma V^T &= (U_1, U_2, U_3)\begin{pmatrix} \Sigma_1 & & \\ & \Sigma_2 & \\ & & 0 \end{pmatrix}(V_1, V_2, V_3)^T \\
&= U_1\Sigma_1 V_1^T + U_2\Sigma_2 V_2^T,
\end{aligned} \tag{3.29}$$

where $\Sigma_1 = \mathrm{diag}\{\sigma_1, \cdots, \sigma_r\}$, $\Sigma_2 = \mathrm{diag}\{\sigma_{r+1}, \cdots, \sigma_{r+q}\}$ with $\sigma_1 \geq \cdots \geq \sigma_r \geq \sigma_{r+1} \geq \cdots \geq \sigma_{r+q} > 0$, and $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices. Let $\bar{x}_k \in P_S(x_k)$. Then, we have the following three inequalities (their proofs are similar to that in [19, Lemma 2.3] and so are omitted here)

$$\|U_1U_1^T f(x_k)\| \leq L_1\|x_k - \bar{x}_k\|; \tag{3.30}$$

$$\|U_2U_2^T f(x_k)\| \leq \frac{3L}{2}\|x_k - \bar{x}\|^2; \tag{3.31}$$

$$\|U_3U_3^T f(x_k)\| \leq \frac{L}{2}\|x_k - \bar{x}_k\|^2. \tag{3.32}$$

Below, we show that (3.27) holds. In fact, it follows from (3.4) and (3.29) that

$$\begin{aligned} f(x_k) + f'(x_k)d_k^* &= \left(\mathbb{I}_m - f'(x_k)f'(x_k)^T\left(f'(x_k)f'(x_k)^T + \lambda_k\mathbb{I}_m\right)^{-1}\right)f(x_k) \\ &= \lambda_k\left(f'(x_k)f'(x_k)^T + \lambda_k\mathbb{I}_m\right)^{-1} f(x_k) \\ &= \lambda_k U_1\left(\Sigma_1^2 + \lambda_k\mathbb{I}_r\right)^{-1}U_1^T f(x_k) + \lambda_k U_2\left(\Sigma_2^2 + \lambda_k\mathbb{I}_q\right)^{-1}U_2^T f(x_k) + U_3U_3^T f(x_k). \end{aligned} \tag{3.33}$$

As $\bar{x} \in \mathbf{B}(x^*, \frac{r_0}{2})$, it follows from (3.28) that $x_k \in \mathbf{B}(x^*, r_0)$. Thus, by Lemma 2.1 and assumption (a) of Lemma 3.2, we have

$$\|\mathrm{diag}(\Sigma_1 - \bar{\Sigma}_1, \Sigma_2, 0)\| \leq \|f'(x_k) - f'(\bar{x})\| \leq L\|x_k - \bar{x}\|.$$

The above inequality implies that

$$\|\Sigma_1 - \bar{\Sigma}_1\| \leq L\|x_k - \bar{x}\| \quad \text{and} \quad \|\Sigma_2\| \leq L\|x_k - \bar{x}\|. \tag{3.34}$$

Observe further from (3.34) that

$$\|\bar{\Sigma}_1^{-1}\| \cdot \|\Sigma_1 - \bar{\Sigma}_1\| \leq \frac{L\|x_k - \bar{x}\|}{\bar{\sigma}_r} \leq \frac{1}{2}.$$

Then, by the perturbation theory, we obtain

$$\|\Sigma_1^{-1}\| \leq \frac{\frac{1}{\bar{\sigma}_r}}{1 - \frac{L\|x_k - \bar{x}\|}{\bar{\sigma}_r}} \leq \frac{2}{\bar{\sigma}_r}.$$

It then follows from the above inequality that

$$\|(\Sigma_1^2 + \lambda_k\mathbb{I}_r)^{-1}\| \leq \|\Sigma_1^{-1}\|^2 \leq \frac{4}{\bar{\sigma}_r^2}. \tag{3.35}$$

Note further that

$$\|(\Sigma_2^2 + \lambda_k\mathbb{I}_q)^{-1}\| \leq \frac{1}{\lambda_k} \quad \text{and} \quad L_1^\delta\|\bar{x}_k - x_k\|^\delta \geq \|f(x_k) - f(\bar{x}_k)\|^\delta = \lambda_k.$$

This, together with (3.35), (3.18), (3.30)-(3.32) and (3.33), implies that

$$\begin{aligned}\|f(x_k)+f'(x_k)d_k^*\| &\le \frac{4\lambda_k}{\bar{\sigma}_r^2}\|U_1U_1^Tf(x_k)\|+\|U_2U_2^Tf(x_k)\|+\|U_3U_3^Tf(x_k)\|\\ &\le \frac{4L_1^{1+\delta}}{\bar{\sigma}_r^2}\|x_k-\bar{x}_k\|^{1+\delta}+2L\|x_k-\bar{x}\|^2\\ &\le \left(\frac{4L_1^{1+\delta}}{\bar{\sigma}_r^2}+2L\right)\|x_k-\bar{x}\|^{1+\delta}, \qquad (3.36)\end{aligned}$$

where the last inequality holds because of $\|x_k-\bar{x}_k\|\le\|x_k-\bar{x}\|<1$. On the other hand, by (3.3), (2.11) and (3.10), we have

$$\|r_k\|\le\theta\|f(x_k)\|^\nu=\theta\|f(x_k)-f(\bar{x}_k)\|^\nu\le\theta L_1^\nu\|\bar{x}_k-x_k\|^\nu\le\theta L_1^\nu\|\bar{x}-x_k\|^\nu. \qquad (3.37)$$

Hence, it follows from (3.8), (3.36) and (3.37) that

$$\|f(x_k)+f'(x_k)d_k\|\le\|f(x_k)+f'(x_k)d_k^*\|+\|r_k\|\le\omega\|x_k-\bar{x}\|^{\min\{\nu,1+\delta\}},$$

where $\omega=\frac{4L_1^{1+\delta}}{\bar{\sigma}_r^2}+2L+\theta L_1^\nu$, and the last inequality holds because of $\|x_k-\bar{x}\|<1$. The proof is completed. □

Lemma 3.4. *Let $0<r_0<1$ and $x^*\in S$. Suppose that assumptions (a) and (b) of Lemma 3.2 hold, and $\beta>\frac{2}{2+\delta}$. Let $\{x_k\}$ be a sequence generated by Algorithm 3.1. Then, there exist $0<q<1$ and $\hat{r}>0$ such that, if $x_0\in\mathbf{B}(x^*,\hat{r})$, then $x_k\in\mathbf{B}(x^*,\frac{r_0}{2})$ for all $k\in\mathbb{N}$, and*

$$\mathrm{d}(x_k,S)\le q^k\hat{r}\quad \textit{for all}\quad k\in\mathbb{N}. \qquad (3.38)$$

Proof. Denote

$$m:=\beta\left(1+\frac{\delta}{2}\right).$$

Note that $\delta\in(0,1]$, we have $\beta>\frac{\delta}{2}$ and $m>1$. Let

$$q:=\left(\frac{1}{2}\right)^{m-1}\quad\text{and}\quad\hat{r}:=\min\left\{\frac{r_0}{2(1+\frac{c}{1-q})},\frac{1}{2c^{\frac{1}{m-1}}}\right\}, \qquad (3.39)$$

where c is the constant given by Lemma 3.2. Since $m-1>0$, we have $0<q<1$. Below, we show by mathematical induction that $x_k\in\mathbf{B}(x^*,\frac{r_0}{2})$ for all $k\in\mathbb{N}$. In fact by (3.11), one has

$$\|x_1-x^*\|\le\|x_0-x^*\|+\|d_0\|\le\|x_0-x^*\|+c\mathrm{d}(x_0,S)\le(1+c)\|x_0-x^*\|\le\frac{r_0}{2},$$

which implies that $x_1\in\mathbf{B}(x^*,\frac{r_0}{2})$. Assume that $x_i\in\mathbf{B}(x^*,\frac{r_0}{2})$ for all $i=1,\cdots,k$. Then, for all $i=1,\cdots,k$, it follows inductively from (3.13) that

$$\mathrm{d}(x_i,S)\le c\mathrm{d}(x_{i-1},S)^m\le\cdots\le c^{\frac{m^i-1}{m-1}}\mathrm{d}(x_0,S)^{m^i}. \qquad (3.40)$$

By the definition of $\hat{r}$ in (3.39) and $x_0 \in \mathbf{B}(x^*, \hat{r})$, we have

$$c^{\frac{m^i-1}{m-1}} \mathrm{d}(x_0, S)^{m^i} = \left(c^{\frac{1}{m-1}} \mathrm{d}(x_0, S)\right)^{m^i-1} \mathrm{d}(x_0, S) \le 2\hat{r} \left(\frac{1}{2}\right)^{m^i}.$$

Combining this with (3.40) yields that

$$\mathrm{d}(x_i, S) \le 2\hat{r} \left(\frac{1}{2}\right)^{m^i} \le 2\hat{r} \left(\frac{1}{2}\right)^{1+(m-1)i} = q^i \hat{r}. \tag{3.41}$$

This, together with (3.11) and (3.39), yields that

$$\|x_{k+1} - x^*\| \le \|x_0 - x^*\| + \sum_{i=0}^{k} \|d_i\| \le \hat{r} + c \sum_{i=0}^{k} \mathrm{d}(x_i, S) \le \hat{r} + c\hat{r} \sum_{i=0}^{k} q^i \le (1 + \frac{c}{1-q})\hat{r} \le \frac{r_0}{2},$$

which implies that $x_{k+1} \in \mathbf{B}(x^*, \frac{r_0}{2})$. Consequently, $x_k \in \mathbf{B}(x^*, \frac{r_0}{2})$ for all $k \in \mathbb{N}$. Furthermore, (3.38) follows directly from (3.41). Thus, the proof is completed. □

Now, we are ready to present the local convergence of Algorithm 3.1 as follows.

Theorem 3.1. *Let $x^* \in S$. Suppose that*

(a) *f' is local Lipschitz continuous around x^*;*

(b) *equation (2.12) has a local error bound of order β around x^*.*

Suppose further that $\beta > \frac{2}{2+\delta}$. Then, there exists $\hat{r} > 0$ such that, for any $x_0 \in \mathbf{B}(x^*, \hat{r})$, the sequence $\{x_k\}$ generated by Algorithm 3.1 with initial point x_0 converges to some solution $\bar{x}$ of (2.12) at order at least $\kappa = \beta \min\{\nu, 1 + \delta\}$. That is, there exists $C > 0$ such that

$$\lim_{k\to\infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|^\kappa} \le C.$$

Consequently, if $\beta = 1, \delta = 1$ and $\nu = 2$, then the convergence rate is at least quadratical.

Proof. By assumptions (a) and (b), there exist $L > 0$, $\mu > 0$ and $0 < r_0 < 1$ such that f' is Lipschitz continuous on $\mathbf{B}(x^*, r_0)$ with modulus L and (2.12) has an error bound of order β on $\mathbf{B}(x^*, r_0)$ with modulus μ. Let $q, \hat{r}$ be given by (3.39). Since $x_0 \in \mathbf{B}(x^*, \hat{r})$, Lemma 3.4 is applicable to conclude that $x_k \in \mathbf{B}(x^*, \frac{r_0}{2})$ for all $k \in \mathbb{N}$, and (3.38) holds. This, together with (3.11), yields that

$$\sum_{k=0}^{\infty} \|d_k\| \le c \sum_{k=0}^{\infty} \mathrm{d}(x_k, S) \le c\hat{r} \sum_{k=0}^{\infty} q^k \le \frac{c\hat{r}}{1-q} < +\infty. \tag{3.42}$$

This means that $\{x_k\}$ is a Cauchy sequence. Suppose that $\{x_k\}$ converges to some point $\bar{x}$. Observe further from (3.42) that

$$\lim_{k\to\infty} \mathrm{d}(x_k, S) = 0 \tag{3.43}$$

and so $\bar{x} \in S$ because S is closed. Hence, $\bar{x}$ is a solution of (2.12). Below, we show that there exists a positive integer N such that for all $k \geq N$,

$$\|d_{k+1}\| \leq 2^{\beta\left(1+\frac{\delta}{2}\right)} c^2 \|d_k\|^{\beta\left(1+\frac{\delta}{2}\right)} \tag{3.44}$$

and

$$\lim_{k\to\infty} \frac{\|\sum_{i=k+1}^{\infty} d_i\|}{\|d_{k+1}\|} = 1. \tag{3.45}$$

Granting this, we have

$$\lim_{k\to\infty} \frac{\|x_{k+1}-\bar{x}\|}{\|x_k-\bar{x}\|^{\beta\left(1+\frac{\delta}{2}\right)}} = \lim_{k\to\infty} \frac{\|\sum_{i=k+1}^{\infty} d_i\|}{\|\sum_{i=k}^{\infty} d_i\|^{\beta\left(1+\frac{\delta}{2}\right)}} = \lim_{k\to\infty} \frac{\|d_{k+1}\|}{\|d_k\|^{\beta\left(1+\frac{\delta}{2}\right)}} \leq 2^{\beta\left(1+\frac{\delta}{2}\right)} c^2, \tag{3.46}$$

which implies that $\{x_k\}$ converges to $\bar{x}$ at order of $m := \beta\left(1+\frac{\delta}{2}\right) > 1$. To proceed, by (3.43), there exists a positive integer N such that for all $k \geq N$,

$$c\mathrm{d}(x_k, S)^{m-1} \leq \frac{1}{2}. \tag{3.47}$$

Fix $k \geq N$. Then, it follows from (3.47) and (3.13) that

$$\mathrm{d}(x_{k+1}, S) \leq c\mathrm{d}(x_k, S)^m \leq \frac{1}{2}\mathrm{d}(x_k, S). \tag{3.48}$$

Observe further that

$$\mathrm{d}(x_k, S) \leq \mathrm{d}(x_{k+1}, S) + \|x_{k+1} - x_k\| = \mathrm{d}(x_{k+1}, S) + \|d_k\|.$$

Combining this with (3.48) yields

$$\mathrm{d}(x_k, S) \leq 2\|d_k\|.$$

This, together with (3.11) and (3.13), gives

$$\|d_{k+1}\| \leq c\mathrm{d}(x_{k+1}, S) \leq c^2\mathrm{d}(x_k, S)^m \leq 2^m c^2 \|d_k\|^m, \tag{3.49}$$

which means that (3.44) holds and $\|d_{k+1}\| = O(\|d_k\|^m)$. By (3.49) and (3.42), there exists a positive integer $N_1 \geq N$, such that for all $k > N_1$,

$$\|d_{k+1}\| \leq M\|d_k\|^{1+\frac{2m-2}{3}}, \tag{3.50}$$

where $M = 2^m c^2 \|d_k\|^{\frac{m-1}{3}} < 1$. Fix $k \geq N_1$. Then, it follows inductively from (3.50) that for each $i \geq 2$,

$$\|d_{k+i}\| \leq M^{3\frac{\left(1+\frac{2m-2}{3}\right)^{i-1}-1}{2m-2}} \|d_{k+1}\|^{\left(1+\frac{2m-2}{3}\right)^{i-1}}.$$

This, together with (3.42), implies that

$$\lim_{k\to\infty}\sum_{i=2}^{\infty}\frac{\|d_{k+i}\|}{\|d_{k+1}\|}\leq\lim_{k\to\infty}\sum_{i=2}^{\infty}(M^{\frac{3}{2m-2}}\|d_{k+1}\|)^{\left(1+\frac{2m-2}{3}\right)^{i-1}-1}=0. \tag{3.51}$$

Observe further that

$$1-\frac{\sum_{i=k+2}^{\infty}\|d_i\|}{\|d_{k+1}\|}\leq\frac{\|\sum_{i=k+1}^{\infty}d_i\|}{\|d_{k+1}\|}\leq 1+\frac{\sum_{i=k+2}^{\infty}\|d_i\|}{\|d_{k+1}\|}. \tag{3.52}$$

Hence, (3.45) follows directly from (3.51) and (3.52).

Below, we shall prove the convergence order. To show this, by Lemma 3.2, we have that (3.27) holds for sufficiently large k. Combining this with (3.11), (3.26) and the fact that $\|x_k-\bar{x}\|<1$ yields that

$$\|d_{k+1}\|\leq c\mathrm{d}(x_{k+1},S)\leq\frac{c}{\mu}\left(\frac{L}{2}\|d_k\|^2+\|f(x_k)+f'(x_k)d_k\|\right)^{\beta}\leq\frac{c\left(2\omega+Lc^2\right)^{\beta}}{\mu 2^{\beta}}\|x_k-\bar{x}\|^{\beta\min\{\nu,1+\delta\}} \tag{3.53}$$

holds for sufficiently large k. Since the sequence $\{x_k\}$ converges to $\bar{x}$ superlinearly by (3.46), we have by [49, Theorem 1.5.2] that

$$\lim_{k\to\infty}\frac{\|d_{k+1}\|}{\|x_{k+1}-\bar{x}\|}=1.$$

This, together with (3.53), gives that

$$\lim_{k\to\infty}\frac{\|x_{k+1}-\bar{x}\|}{\|x_k-\bar{x}\|^{\kappa}}=\lim_{k\to\infty}\frac{\|d_{k+1}\|}{\|x_k-\bar{x}\|^{\kappa}}\leq\frac{c\left(2\omega+Lc^2\right)^{\beta}}{\mu 2^{\beta}}=C,$$

which implies that $\{x_k\}$ converges to $\bar{x}$ at order of $\kappa=\beta\min\{\nu,1+\delta\}$. This completes the proof. $\square$

4 A global version of the modified inexact LMM and its global convergence analysis

This section is devoted to presenting a globalization strategy for the modified inexact LMM with line-search scheme and establishing its global convergence. Firstly, we present three types of the typical line-search rules for selecting the sequence of step sizes $\{\alpha_k\}$ for Algorithm 4.1, which have been widely used in the literature; see, e.g., [2, 30, 44].

Definition 4.1. *Let $\xi\in(0,1)$, $\sigma_1\in(0,\frac{1}{2})$ and let $\sigma_2\in(\sigma_1,1)$. Given $k\geq 0$, x_k and d_k, a stepsize $\alpha_k\in(0,+\infty)$ is said to satisfy*

(a) *the Armijo rule if*

$$\phi(x_k + \alpha_k d_k) \le \phi(x_k) + \sigma_1 \alpha_k \nabla\phi(x_k)^T d_k \tag{4.1}$$

and

$$\alpha_k := \max\{\xi^i : i \in \mathbb{N},\ \phi(x_k + \xi^i d_k) \le \phi(x_k) + \sigma_1 \xi^i \nabla\phi(x_k)^T d_k\}. \tag{4.2}$$

(b) *the Goldstein rule if* (4.1) *holds and*

$$\phi(x_k) + (1-\sigma_1)\alpha_k \nabla\phi(x_k)^T d_k \le \phi(x_k + \alpha_k d_k).$$

(c) *the Wolfe rule if* (4.1) *holds and*

$$\nabla\phi(x_k + \alpha_k d_k)^T d_k \ge \sigma_2 \nabla\phi(x_k)^T d_k. \tag{4.3}$$

The modified inexact LMM with line-search scheme is formulated as follows.

Algorithm 4.1

Step 0 Choose a starting point $x_0 \in \mathbb{R}^n$, parameters $\delta \in (0,1]$, $\gamma, \xi \in (0,1)$, $\sigma_1 \in (0, \frac{1}{2})$, $\sigma_2 \in (\sigma_1, 1)$, $\rho \in (0,+\infty)$ and $\nu \in \left[1+\frac{\delta}{2}, 2\right]$, and a sequence $\{\theta_k\} \subseteq \mathbb{R}_+$ such that $\theta = \sup_{k\in\mathbb{N}} \theta_k < 1$. Set $k := 0$.

Step 1 Generate d_k by Steps 1 and 2 in Algorithm 3.1.

Step 2 If d_k satisfies

$$\|f(x_k + d_k)\| \le \gamma \|f(x_k)\|, \tag{4.4}$$

then set $x_{k+1} := x_k + d_k$, and go to step 4. Otherwise, go to step 3.

Step 3 If d_k satisfies

$$\nabla\phi(x_k)^T d_k \le -\rho \|\nabla\phi(x_k)\|^2, \tag{4.5}$$

set $d_k := d_k$; otherwise set $d_k := -\nabla\phi(x_k)$. Select the step size $\alpha_k \in (0,+\infty)$ satisfying the Armijo rule, or the Goldstein rule, or the Wolfe rule. Set $x_{k+1} := x_k + \alpha_k d_k$.

Step 4 Set $k := k+1$. Go to Step 1.

Remark 4.1. *It follows from Steps 2 and 3 that the sequence $\{\|f(x_k)\|\}$ is monotonically decreasing and bounded from below, and thus convergent.*

Remark 4.2. *The condition* (4.5) *in Step 3 of Algorithm 4.1 is different from*

$$\nabla\phi(x_k)^T d_k \le -\rho\|d_k\|^p \quad \text{with some } p > 0,$$

which was used in [12]. With the aid of the residual control $\|\tilde{r}_k\| \le \eta\|\nabla\phi(x_k)\|$ in [12], the authors proved that there exist $\gamma_1, \gamma_2 \in (0,+\infty)$ such that

$$\gamma_1\|\nabla\phi(x_{k_i})\| \le \|d_{k_i}\| \le \gamma_2\|\nabla\phi(x_{k_i})\| \tag{4.6}$$

on any convergent subsequence $\{x_{k_i}\}$. On the contrary, (4.6) *does not hold in general in this paper due to our residual control. This is the reason why we choose* (4.5).

Now, we are ready to establish the global convergence result of Algorithm 4.1. Recall that f is Fréchet differentiable and its Fréchet derivative is continuous.

Proposition 4.1. *Let $\{x_k\}$ be a sequence generated by Algorithm 4.1. Suppose that $\{x_k\}$ has an accumulation point x^*. Then, x^* is a stationary point of ϕ.*

Proof. As x^* is an accumulation point of $\{x_k\}$, there exists a subsequence $\{x_{k_i}\}$ such that $\lim_{i\to\infty} x_{k_i} = x^*$. Let

$$K_1 = \{i|\ \|f(x_{k_i} + d_{k_i})\| \leq \gamma\|f(x_{k_i})\|\}.$$

We divide the proof into two cases.

Case 1. K_1 is infinite. Then, there exists a subsequence of $\{x_{k_i}\}$, denoted by itself, such that

$$\|f(x_{k_i} + d_{k_i})\| \leq \gamma\|f(x_{k_i})\| \quad \text{for all } i \in \mathbb{N}.$$

This, together with Remark 4.1, implies that for each $k_i \geq 0$, there is a positive integer j_i such that

$$\|f(x_{k_i})\| \leq \gamma^{j_i}\|f(x_0)\| \quad \text{and} \quad \lim_{i\to\infty} j_i = +\infty.$$

Hence, $\lim_{i\to\infty} \|f(x_{k_i})\| = 0$. Since f is continuous, we have $f(x^*) = 0$. Thus, x^* is a stationary point of ϕ.

Case 2. K_1 is finite. We assume without loss of generality that

$$\|f(x_{k_i} + d_{k_i})\| > \gamma\|f(x_{k_i})\| \quad \text{for all } i \in \mathbb{N}.$$

In view of Algorithm 4.1, the line-search step is executed for each k_i. Hence, we can apply [6, Propositions 1.2.1 and 1.2.2] to conclude that x^* is a stationary point of ϕ. □

Corollary 4.1. *Let $\{x_k\}$ be a sequence generated by Algorithm 4.1. Suppose that the level set $\mathcal{L} := \{x \in \mathbb{R}^n|\phi(x) \leq \phi(x_0)\}$ is bounded. Then, $\{x_k\}$ has an accumulation point which is a stationary point of ϕ.*

Proof. It follows from Remark 4.1 that the sequence $\{x_k\} \subseteq \mathcal{L}$. Since the level set $\mathcal{L}$ is bounded (due to assumption), we get that the sequence $\{x_k\}$ has an accumulation point. Thus, it follows from Proposition 4.1 that the accumulation point is a stationary point of ϕ. □

Theorem 4.1. *Let $\{x_k\}$ be a sequence generated by Algorithm 4.1. Suppose that $\beta > \sqrt{\frac{2}{2+\delta}}$ and $\{x_k\}$ has an accumulation point x^* such that*

(a) *$f(x^*) = 0$ and f' is local Lipschitz continuous around x^*;*

(b) *equation (2.12) has a local error bound of order β around x^*.*

Then, $\{x_k\}$ converges to x^* at order at least $\kappa = \beta\min\{\nu, 1+\delta\}$. Consequently, if $\beta = 1, \delta = 1$ and $\nu = 2$, then the convergence rate is at least quadratical.

Proof. Let $\{x_{k_i}\}$ be a subsequence of $\{x_k\}$ such that $\lim_{i\to\infty} x_{k_i} = x^*$. Since f is continuous, we obtain that $\lim_{i\to\infty} f(x_{k_i}) = 0$. Hence, there exists a positive integer N such that

$$\|x_{k_i} - x^*\| \le \hat{r} \quad \text{and} \quad \|f(x_{k_i})\| \le \left(\frac{\gamma\mu^{\beta(1+\delta/2)}}{L_1 c}\right)^{\frac{1}{\beta^2(1+\delta/2)-1}} \qquad \forall i \ge N, \tag{4.7}$$

where $\hat{r}, L_1$ are given by Lemma 3.4 and (2.11), respectively, while c is given by Lemma 3.2. Set $k = k_N$. By (2.11) and (3.13), we have

$$\|f(x_k + d_k)\| \le L_1 \mathrm{d}(x_k + d_k, S) \le L_1 c \mathrm{d}(x_k, S)^{\beta(1+\frac{\delta}{2})}.$$

This, together with (2.13), gives that

$$\|f(x_k + d_k)\| \le L_1 c \mu^{-\beta(1+\frac{\delta}{2})} \|f(x_k)\|^{\beta^2(1+\frac{\delta}{2})-1} \|f(x_k)\| \le \gamma \|f(x_k)\| \tag{4.8}$$

(by (4.7)); hence (4.4) holds for $k = k_N$. Furthermore, since $\gamma \in (0,1)$, it follows from (4.8) and (4.7) that

$$\|f(x_{k+1})\| \le \|f(x_k)\| \le \left(\frac{\gamma\mu^{\beta(1+\delta/2)}}{L_1 c}\right)^{\frac{1}{\beta^2(1+\delta/2)-1}}.$$

Thus, inductively, with the same arguments as we did for (4.8), we can obtain that (4.4) holds for $k = k_N + 1$. Consequently, by mathematical induction, we conclude that (4.4) holds for all $k \ge k_N$. Therefore, Algorithm 4.1 is reduced to Algorithm 3.1 for all $k \ge k_N$. Thus, Theorem 3.1 is applicable to concluding that the sequence $\{x_k\}$ converges to x^* superlinearly when $\delta \in (0,1)$ and quadratically when $\delta = 1$, $\beta = 1$ and $\nu = 2$. This completes the proof. □

Let $\beta = 1$. Then Corollary 4.2 follows directly from Theorem 4.1, which extends the corresponding results in [12, Theorem 3.1] and [19, Theorem 3.1].

Corollary 4.2. *Let $\{x_k\}$ be a sequence generated by Algorithm 4.1. Suppose that $\{x_k\}$ has an accumulation point x^* such that*

(a) *$f(x^*) = 0$ and f' is local Lipschitz continuous around x^*;*

(b) *equation* (2.12) *has a local error bound around x^*.*

Then, $\{x_k\}$ converges to x^* at order at least $\kappa = \min\{\nu, 1+\delta\}$. Consequently, if $\delta = 1$ and $\nu = 2$, then the convergence rate is at least quadratical.

5 Numerical examples

In this section, we conduct some numerical experiments to show the efficiency of the globalized version of the modified inexact LMM, compared with the globalized version of classical inexact LMM proposed in [12]. In both algorithms, the linear equations are solved by the conjugate gradient (CG) method (see [44]) or Cholesky factorization, and the Armijo, Wolfe or Goldstein line-search schemes are adopted. All the tests were implemented in MATLAB R2016a on a Lenovo PC with Intel(R) Core(TM) i5-3210 CPU @ 2.5 GHz.

In the numerical experiments, we consider some medium/large dimension test problems with $f := (f_1,\ f_2,\ \dots,\ f_m)^T$ defined as follows, which are taken from [12, 42] (with some modifications). For $x \in \mathbb{R}^n$, we use $x^{(i)}$ to denote the i-th component of x. As pointed out in [12], we can check that the solution set of each of these problems is not locally unique but (2.12) has a local error bound in a neighborhood of each solution. Note that problem (P4) is underdetermined and also row rank deficient.

(P1) $f : \mathbb{R}^{2m} \to \mathbb{R}^m$ where

$$f_i(x) := x^{(i)}x^{(m+i)} - \sqrt{i}, \quad i = 1, \cdots, m.$$

(P2) $f : \mathbb{R}^{2m} \to \mathbb{R}^m$ where

$$f_i(x) := \left(3 - 2x^{(2i-1)}\right) x^{(2i-1)} - x^{(2i-2)} - 2x^{(2i)} + 1, \quad i = 1, \cdots, m.$$

(P3) $f : \mathbb{R}^{3m} \to \mathbb{R}^m$ where

$$f_i(x) := x^{(i)}x^{(m+i)}x^{(2m+i)} - \sqrt[3]{i}, \quad i = 1, \cdots, m.$$

(P4) $f : \mathbb{R}^{2m} \to \mathbb{R}^m$ where

$$f_i(x) := \begin{cases} \sqrt{i}\exp\left(\left(\sum_{j=2i-1}^{2i+2} x^{(j)}\right)/m\right) - \sqrt{i}, & \mathrm{mod}(i,2) = 1; \\ \sqrt{i}\left(\sum_{j=2i-3}^{2i} x^{(j)}\right)\left(\sum_{j=2i-3}^{2i} x^{(j)} - 1\right), & \mathrm{mod}(i,2) = 0. \end{cases} \quad i = 1, \cdots, m.$$

For each problem mentioned above, we consider m = 1000, 2500, and 4000, and choose the following initial points:

(I1) $x_{01} = \left(10^{-5}, -m/2,\ \cdots,\ 10^{-5}, -m/2\right)^T$ in problem (P1);

(I2) $x_{02} = (m/100,\ \cdots,\ m/100)^T$ in problem (P2);

(I3) $x_{03} = (-m/2,\ \cdots,\ -m/2)^T$ in problem (P3);

(I4) $x_{04} = (-m/2,\ \cdots,\ -m/2)^T$ in problem (P4).

In the numerical experiments, we mainly compare the proposed Algorithm 4.1 with the classical inexact LMM proposed in [12], which is stated as follows.

Algorithm 5.1

Step 0 Same as Step 0 in Algorithm 4.1.

Step 1 Generate d_k by Steps 1 and 2 in Algorithm ILM.

Step 2 Same as Step 2 in Algorithm 4.1

Step 3 If

$$\nabla\phi(x_k)^T d_k \le -\rho\|d_k\|^p,$$

is not satisfied, set $d_k := -\nabla\phi(x_k)$. Select the step size $\alpha_k \in (0, +\infty)$ satisfying the Armijo rule, or the Goldstein rule, or the Wolfe rule. Set $x_{k+1} := x_k + \alpha_k d_k$.

Step 4 Same as Step 4 in Algorithm 4.1.

In the implementation of Algorithms 4.1 and 5.1, we set the parameters as follows: $\sigma_1 = 0.6$ (in the Armijo and Wolfe schemes) or 0.2 (in the Goldstein scheme), $\sigma_2 = 0.9$, $\xi = 0.7$, $\gamma = 0.8$, $\delta = 1.0$, $\eta = 0.8$, $\rho = 2.0$, $\tau = 2.0$, $p = 2.0$, $\theta = 0.8$, $\zeta = 0.001$, and regularization parameters $\lambda_k = \min\left\{\|f(x_k)\|^\delta, \zeta\right\}$. In the Wolfe and Goldstein line-search schemes, we interpolate by using bisection to find a trial step length α_k in an interval. To compare with the two inexact LMMs, we take the same stopping criterion for both (3.3) in Algorithm 4.1 and (1.5) in Algorithm 5.1 as

$$\|r_k\| \le \epsilon_k = \min\left\{\theta\|f(x_k)\|, \theta\|f(x_k)\|^2, 0.001\sqrt{n}\right\}. \tag{5.1}$$

Finally, the stopping criterion of the outer loops in both Algorithms 4.1 and 5.1 is set as

$$\|f(x_k)\| \le 10^{-8}\sqrt{n}.$$

We first report in Tables 1-5 the experimental results when adopting the CG method to solve the linear equations, where i.p. denotes the initial point, N_o, N_i and N_l denote the numbers of outer iterations, inner iterations (of the CG method) and the line-search iterations, respectively. The CPU time in seconds is denoted by t and $\|f(x_k)\|$ lists its values at the last three iterations of each algorithm.

As mentioned above, we use the CG method to solve the revolved linear equations (1.7) and (1.3) in Algorithms 4.1 and 5.1, respectively. To reduce the computational costs, the calculation of $f'(x_k)f'(x_k)^T u$ in (1.7) is implemented by the successive matrix-vector multiplication $w = f'(x_k)^T u$ and $f'(x_k)w$, that is, the matrix $M_k = f'(x_k)f'(x_k)^T$ is not explicitly calculated throughout the numerical experiments, for problems (P1)-(P3). The similar computation strategy is applied in the CG method for solving (1.3) in Algorithm 5.1. From Tables 1-3, we observe that Algorithm 4.1 requires less CPU time and less inner iteration numbers to approach the solution than Algorithm 5.1 does for problems (P1)-(P3).

In general, as illustrated in Algorithms 4.1 and 5.1 for problems (P1)-(P3), when conducting the CG iterations, the successive matrix-vector multiplication requires less CPU time than explicitly forming the coefficient matrix (though we do not list the corresponding results for problems (P1)-(P3) with explicitly forming the matrix). However, in the case when the inner iterations is much more than the outer iterations (as in Algorithms 4.1 and 5.1 for problem (P4)), Algorithm 4.1 with explicitly forming matrix strategy may be more efficient than Algorithm 4.1 with successive matrix-vector multiplication or Algorithm 5.1 with either of these two strategies (see Tables 4 and 5). This is because, at the step for solving the subproblem, the additional cost caused by explicitly forming the coefficient matrix in Algorithm 4.1 with explicitly forming matrix strategy is less than the extra one due to the larger subproblems in Algorithm 4.1 with successive matrix-vector multiplications or in Algorithm 5.1 (compared with the explicitly forming matrix strategy in Algorithm 4.1).

In conclusion, from the preliminary numerical results, we can see that Algorithm 4.1 (as well as Algorithm 5.1) converges quadratically to a solution of NLSP, and that Algorithm 4.1 outperforms Algorithm 5.1 as in costing less CPU time to approach the solution for the underdetermined case.

Table 1: Results for problem (P1) by applying CG method.

$m \times n$	i.p.	alg./linesearch	N_o	N_i	N_l	t	$\|\|f(x_k)\|\|$
		4.1/no	13	316	0	0.9	3.2e-03, 8.2e-06, 4.4e-11
1000×2000	x_{01}	5.1/Armijo	81	4874	69	13.3	3.4e-02, 3.2e-04, 7.0e-08
		5.1/Wolfe	15	401	3	1.6	1.5e-02, 4.5e-05, 6.4e-10
		5.1/Goldstein	126	5368	115	16.1	2.6e-02, 1.9e-04, 2.5e-08
		4.1/no	14	406	0	7.0	2.6e-02, 4.2e-04, 1.4e-07
2500×5000	x_{01}	5.1/Armijo	17	796	2	14.8	1.4e-02, 3.0e-05, 2.6e-10
		5.1/Wolfe	17	796	2	13.9	1.8e-02, 5.7e-05, 1.0e-09
		5.1/Goldstein	17	783	2	13.6	4.3e-02, 5.1e-04, 1.5e-07
		4.1/no	15	495	0	21.0	3.8e-03, 9.8e-06, 7.7e-11
4000×8000	x_{01}	5.1/Armijo	17	906	2	38.3	2.2e-02, 7.7e-05, 1.9e-09
		5.1/Wolfe	18	1197	2	49.4	1.0e-02, 3.3e-05, 7.1e-10
		5.1/Goldstein	18	1200	2	47.7	1.5e-02, 5.8e-05, 1.9e-09

Table 2: Results for problem (P2) by applying CG method.

$m \times n$	i.p.	alg./linesearch	N_o	N_i	N_l	t	$\|\|f(x_k)\|\|$
		4.1/Armijo	10	56	2	0.9	2.2e-03, 3.6e-06, 7.5e-12
		4.1/Wolfe	10	45	1	1.0	5.1e-03, 5.3e-06, 9.7e-12
		4.1/Goldstein	10	59	1	0.9	2.0e-02, 1.9e-04, 2.4e-08
1000×2000	x_{02}	5.1/Armijo	11	83	2	1.2	1.7e-03, 1.3e-06, 1.4e-12
		5.1/Wolfe	10	54	2	1.2	3.0e-02, 1.0e-04, 3.6e-09
		5.1/Goldstein	11	73	1	1.2	4.4e-02, 6.0e-04, 2.4e-07
		4.1/Armijo	18	88	6	9.9	3.1e-02, 7.1e-04, 2.6e-07
		4.1/Wolfe	11	43	2	6.9	3.0e-02, 2.6e-04, 2.6e-08
		4.1/Goldstein	20	97	5	10.9	1.9e-03, 9.5e-07, 4.3e-13
2500×5000	x_{02}	5.1/Armijo	18	110	6	12.0	8.1e-02, 7.1e-04, 8.9e-08
		5.1/Wolfe	21	139	9	18.1	4.5e-03, 5.6e-06, 9.3e-12
		5.1/Goldstein	15	76	2	9.8	4.3e-03 1.4e-06 3.8e-13
		4.1/Armijo	16	74	4	22.6	5.6e-03, 1.7e-05, 1.9e-10
		4.1/Wolfe	17	89	5	29.7	1.2e-03, 1.2e-06, 3.6e-13
		4.1/Goldstein	21	108	10	29.0	4.2e-02, 1.2e-03, 7.2e-07
4000×8000	x_{02}	5.1/Armijo	25	237	11	43.9	3.0e-02, 3.2e-04, 4.4e-08
		5.1/Wolfe	35	345	22	86.8	2.0e-02 5.6e-05 1.0e-09
		5.1/Goldstein	29	265	17	51.5	3.3e-02, 2.5e-04, 1.5e-08

Table 3: Results for problem (P3) by applying CG method.

$m \times n$	i.p.	alg./linesearch	N_o	N_i	N_l	t	$\|\|f(x_k)\|\|$
		4.1/Armijo	22	144	2	1.9	7.1e-03, 3.8e-05, 8.8e-10
		4.1/Wolfe	22	147	2	2.1	1.1e-03, 9.1e-07, 4.6e-13
		4.1/Goldstein	63	2196	44	14.9	2.2e-03, 3.2e-06, 6.8e-12
1000×3000	x_{03}	5.1/Armijo	82	10864	64	49.5	2.2e-02, 2.8e-05, 3.4e-10
		5.1/Wolfe	44	4345	25	20.2	5.2e-03, 8.4e-06, 3.6e-11
		5.1/Goldstein				> 600	
		4.1/Armijo	24	139	2	9.8	2.6e-02, 5.3e-04, 1.5e-07
		4.1/Wolfe				> 600	
		4.1/Goldstein	24	147	2	8.9	2.1e-02, 3.7e-04, 8.8e-08
2500×7500	x_{03}	5.1/Armijo	48	9075	27	189.3	3.3e-02, 2.9e-04, 3.5e-08
		5.1/Wolfe	50	9792	29	207.2	7.0e-03, 1.3e-05, 5.8e-11
		5.1/Goldstein				> 600	
		4.1/Armijo	25	178	2	25.6	1.0e-02, 8.2e-05, 5.0e-09
		4.1/Wolfe	25	158	2	26.1	7.6e-03, 4.3e-05, 1.2e-09
		4.1/Goldstein	26	231	2	28.7	9.0e-03, 5.0e-05, 1.5e-09
4000×12000	x_{03}	5.1/Armijo	31	1064	9	85.5	2.6e-02, 1.3e-04, 9.6e-09
		5.1/Wolfe	27	524	5	57.6	1.0e-01, 2.1e-03, 1.0e-06
		5.1/Goldstein				> 600	

Table 4: Results for problem (P4) by explicitly forming the matrix.

$m \times n$	i.p.	alg./linesearch	N_o	N_i	N_l	t	$\|\|f(x_k)\|\|$
		4.1/no	17	2469	0	2.9	1.5e-02, 1.5e-04, 1.6e-08
1000×2000	x_{04}	5.1/no	17	2072	0	7.9	1.3e-03, 1.0e-06, 8.9e-13
		4.1/no	19	4601	0	38.7	3.4e-03, 9.4e-06, 6.7e-11
2500×5000	x_{04}	5.1/no	18	4085	0	97.2	4.7e-03, 4.9e-06, 1.9e-11
		4.1/no	20	6039	0	146.4	5.5e-03, 2.1e-05, 3.4e-10
4000×8000	x_{04}	5.1/no	19	6017	0	390.3	4.5e-03, 8.2e-06, 9.0e-11

Table 5: Results for problem (P4) by successive matrix-vector multiplications.

$m \times n$	i.p.	alg./linesearch	N_o	N_i	N_l	t	$\|\|f(x_k)\|\|$
		4.1/no	17	2470	0	5.3	1.5e-02, 1.5e-04, 1.6e-08
1000×2000	x_{04}	5.1/no	17	1998	0	4.8	1.3e-03, 1.0e-06, 8.8e-13
		4.1/no	19	4613	0	56.5	3.4e-03, 9.4e-06, 6.7e-11
2500×5000	x_{04}	5.1/no	18	3512	0	46.2	4.7e-03, 4.9e-06, 1.8e-11
		4.1/no	20	6039	0	189.7	5.5e-03, 2.1e-05, 3.4e-10
4000×8000	x_{04}	5.1/no	19	4685	0	152.6	4.5e-03, 8.2e-06, 5.5e-11

To conclude this section, we illustrate the experimental results of Algorithms 4.1 and 5.1 when adopting the Cholesky factorization to solve the linear equations (1.7) and (1.3), respectively. In fact, because of the special structure of the symmetric matrices involved in (1.7), it is not required to compute directly the matrix-matrix multiplication $f'(x_k)f'(x_k)^T$ and the Cholesky factorization of $A_k = f'(x_k)f'(x_k)^T + \lambda_k I_m$. Alternatively, we do the following QR factorization:

$$\begin{pmatrix} f'(x_k)^T \\ \sqrt{\lambda_k} I_m \end{pmatrix} = Q_k \begin{pmatrix} R_k \\ 0 \end{pmatrix},$$

where Q_k is orthogonal and R_k is upper triangular. Clearly, $R_k^T R_k = f'(x_k)f'(x_k)^T + \lambda_k I_m$, which is Cholesky factorization of A_k. Hence, we solve linear equations (1.7) by solving $R_k^T y = -f(x_k)$ and $R_k s = y$ sequentially. Similarly for Algorithm 5.1, the QR factorization for the symmetric matrices involved in (1.3) is

$$\begin{pmatrix} f'(x_k) \\ \sqrt{\lambda_k} I_n \end{pmatrix} = \bar{Q}_k \begin{pmatrix} \bar{R}_k \\ 0 \end{pmatrix},$$

and one obtains its Cholesky factorization as $\bar{R}_k^T \bar{R}_k = f'(x_k)^T f'(x_k) + \lambda_k I_n$. The numerical results of Algorithms 4.1 and 5.1 when adopting the Cholesky factorization for problems (P1)-(P4) are listed in Tables 6-9, which demonstrates that, in general, Algorithm 4.1 outperforms

Algorithm 5.1 as in costing less CPU time for the underdetermined case. Furthermore, we observe from the numerical results that the CG method is generally more efficient than the Cholesky factorization technique in solving the linear equations.

Table 6: Results for problem (P1) by applying Cholesky factorization.

$m \times n$	i.p.	alg./linesearch	N_o	N_i	N_l	t	$\|\|f(x_k)\|\|$
		4.1/no	16	0	0	9.7	1.3e-01, 3.8e-05, 3.9e-12
1000×2000	x_{01}	5.1/no	12	0	0	21.3	5.6e-02, 3.6e-04, 6.6e-08
		4.1/no	17	0	0	118.8	1.7e+00, 2.5e-03, 5.7e-09
2500×5000	x_{01}	5.1/no	14	0	0	314.3	2.5e-03, 1.8e-06, 1.8e-12
		4.1/no	18	0	0	497.7	5.5e-01, 1.5e-04, 1.5e-11
4000×8000	x_{01}	5.1/no	14	0	0	1349.5	5.6e-02, 3.5e-04, 6.6e-08

Table 7: Results for problem (P2) by applying Cholesky factorization.

$m \times n$	i.p.	alg./linesearch	N_o	N_i	N_l	t	$\|\|f(x_k)\|\|$
		4.1/Armijo	9	0	2	7.3	7.3e-02, 4.4e-04, 5.1e-08
		4.1/Wolfe	9	0	1	7.7	8.0e-02, 4.5e-04, 4.8e-08
		4.1/Goldstein	10	0	1	7.7	1.6e-03, 4.5e-07, 6.5e-14
1000×2000	x_{02}	5.1/Armijo	10	0	2	18.1	7.0e-02, 3.9e-04, 3.1e-08
		5.1/Wolfe	10	0	2	19.2	2.8e-02, 5.5e-05, 7.7e-10
		5.1/Goldstein	11	0	1	20.1	4.2e-02, 1.9e-05, 7.7e-11
		4.1/Armijo	18	0	6	171.7	2.9e-02, 3.5e-05, 8.2e-11
		4.1/Wolfe	11	0	2	102.7	2.7e-02, 7.9e-05, 1.2e-09
		4.1/Goldstein	18	0	5	170.4	1.3e-01, 9.2e-04, 1.9e-07
2500×5000	x_{02}	5.1/Armijo	18	0	6	411.2	7.9e-02, 3.9e-04, 1.5e-08
		5.1/Wolfe	20	0	9	451.5	1.1e-01, 9.4e-04, 2.0e-07
		5.1/Goldstein	15	0	2	338.7	3.1e-03, 1.3e-06, 4.3e-13
		4.1/Armijo	15	0	4	502.4	8.7e-02, 8.0e-04, 1.8e-07
		4.1/Wolfe	16	0	5	538.7	3.1e-03, 8.9e-07, 1.1e-13
		4.1/Goldstein	21	0	10	710.0	4.6e-03, 3.2e-06, 2.8e-12
4000×8000	x_{02}	5.1/Armijo	25	0	11	2271.4	2.5e-02, 6.3e-05, 8.8e-10
		5.1/Wolfe	26	0	14	2373.0	1.9e-01, 1.0e-03, 2.7e-07
		5.1/Goldstein	23	0	11	2046.7	2.8e-02, 7.1e-05, 1.9e-09

Table 8: Results for problem (P3) by applying the Cholesky factorization.

$m \times n$	i.p.	alg./linesearch	N_o	N_i	N_l	t	$\|\|f(x_k)\|\|$
		4.1/Armijo	21	0	2	18.9	8.7e-02, 3.4e-05, 3.3e-10
		4.1/Wolfe	20	0	2	19.3	9.8e-01, 2.9e-03, 1.0e-07
		4.1/Goldstein	61	0	42	58.8	1.5e-03, 1.2e-06, 1.0e-12
1000×3000	x_{03}	5.1/Armijo	20	0	2	106.0	7.6e-02, 1.6e-04, 7.4e-10
		5.1/Wolfe	19	0	2	101.2	8.9e-02, 1.6e-04, 7.1e-10
		5.1/Goldstein	21	0	1	112.0	6.8e-03, 1.3e-06, 5.2e-14
		4.1/Armijo	23	0	2	237.5	7.9e-01, 1.0e-03, 1.3e-08
		4.1/Wolfe				> 1000	
		4.1/Goldstein	23	0	2	233.4	6.3e-01, 1.0e-03, 1.3e-08
2500×7500	x_{03}	5.1/no	7	0	0	513.6	3.6e+05, 4.0e-01, 3.9e-14
		4.1/Armijo	24	0	2	899.7	7.2e-02, 4.1e-04, 1.1e-07
		4.1/Wolfe	23	0	2	891.4	1.7e+00, 3.7e-03, 1.2e-07
		4.1/Goldstein	25	0	2	961.8	9.7e-02, 4.7e-04, 6.0e-08
4000×12000	x_{03}	5.1/no	2	0	0	597.5	5.0e+11, 2.7e-05, 5.5e-14

Table 9: Results for problem (P4) by applying Cholesky factorization.

$m \times n$	i.p.	alg./linesearch	N_o	N_i	N_l	t	$\|\|f(x_k)\|\|$
		4.1/no	16	0	0	9.7	1.3e-01, 3.8e-05, 3.9e-12
1000×2000	x_{04}	5.1/no	16	0	0	27.4	1.3e-01, 3.8e-05, 3.9e-12
		4.1/no	17	0	0	118.8	1.7e+00, 2.5e-03, 5.7e-09
2500×5000	x_{04}	5.1/no	17	0	0	392.8	1.7e+00, 2.5e-03, 5.7e-09
		4.1/no	18	0	0	497.7	5.5e-01, 1.5e-04, 1.5e-11
4000×8000	x_{04}	5.1/no	18	0	0	1651.2	5.5e-01, 1.5e-04, 6.0e-11

Remark 5.1. *We only reported the numerical results for the proposed method without the line-search scheme in Tables 1, 4-6, 9 because for these problems, Step 2 of Algorithms 4.1 and/or 5.1 is satisfied for each k, and thus the line-search in Step 3 is skipped automatically. Actually, it depends on the specific problem structure and the certain algorithm.*

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