# Convergence of a Ulm-like method for square inverse singular value problems with multiple and zero singular values

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**Abstract.** An interesting problem was raised in [SIAM J. Matrix Anal. Appl. 32 (2011), 412–429]: whether the Ulm-like method and its convergence result can be extended to the cases of multiple and zero singular values. In this paper, we study the convergence of a Ulm-like method for solving the square inverse singular value problem with multiple and zero singular values. Under the nonsingularity assumption in terms of the relative generalized Jacobian matrices, a convergence analysis for the multiple and zero case is provided and the quadratical convergence property is proved. Moreover, numerical experiments are given in the last section to demonstrate our theoretic results.

**Keywords.** Inverse singular value problem, Newton's method, Ulm-like method

**AMS subject classification.** 65F18, 65F10, 15A18

## 1 Introduction

The inverse singular value problem (ISVP) arises in different applications such as the determination of mass distributions, orbital mechanics, irrigation theory, computed tomography, circuit theory, etc. [12, 13, 15, 16, 17, 19, 21, 26, 27, 30]. In the present paper, we consider the following special kind of ISVP. Let $p$ and $q$ be two positive integers. Let $\mathbb{R}^p$ denote the $p$-dimensional Euclidean space and $\mathbb{R}^{p\times q}$ be the set of all real $p \times q$ matrices. Let $m$ and $n$ be two positive integers such that $m \geq n$. Let $\{A_i\}_{i=0}^n \subset \mathbb{R}^{m\times n}$. Given $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$, we define

$$A(\mathbf{c}) := A_0 + \sum_{i=1}^{n} c_i A_i \tag{1.1}$$

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and denote its singular values by $\{\sigma_i(\mathbf{c})\}_{i=1}^n$ with the order $\sigma_1(\mathbf{c}) \geq \sigma_2(\mathbf{c}) \geq \cdots \geq \sigma_n(\mathbf{c}) \geq 0$. The ISVP considered here is, for $n$ given real numbers $\{\sigma_i^*\}_{i=1}^n$ ordering with

$$\sigma_1^* \geq \sigma_2^* \geq \cdots \geq \sigma_n^* \geq 0,$$

to find a vector $\mathbf{c}^* \in \mathbb{R}^n$ such that $\{\sigma_i^*\}_{i=1}^n$ are exactly the singular values of $A(\mathbf{c}^*)$, i.e.,

$$\sigma_i(\mathbf{c}^*) = \sigma_i^*, \quad \text{for each } i = 1, 2, \ldots, n. \tag{1.2}$$

The vector $\mathbf{c}^*$ is called a solution of the ISVP (1.2). This type of ISVP was originally proposed by Chu [5] in 1992 and was further studied in [2, 3, 5, 14, 24, 28]. In the case when $m = n$, we call the problem is square. Obviously, if $\{A_i\}_{i=0}^n$ are symmetric, the square inverse eigenvalue problem is reduced to the inverse eigenvalue problem (IEP) which arises in a variety of applications and was studied extensively in [1, 4, 6, 7, 10, 22, 29].

Define the function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathbf{f}(\mathbf{c}) := (\sigma_1(\mathbf{c}) - \sigma_1^*,\ \sigma_2(\mathbf{c}) - \sigma_2^*,\ \ldots,\ \sigma_n(\mathbf{c}) - \sigma_n^*)^T, \quad \text{for any } \mathbf{c} \in \mathbb{R}^n. \tag{1.3}$$

Then, as noted in [2, 3, 5, 14, 24, 28], solving the ISVP (1.2) is equivalent to finding a solution $\mathbf{c}^* \in \mathbb{R}^n$ of the nonlinear equation $\mathbf{f}(\mathbf{c}) = \mathbf{0}$. Recall that, in the case when the given singular values are distinct and positive, i.e.,

$$\sigma_1^* > \sigma_2^* > \cdots > \sigma_n^* > 0, \tag{1.4}$$

there exists a neighborhood of $\mathbf{c}^*$ where the function $\mathbf{f}$ is differentiable around the solution $\mathbf{c}^*$ and the singular vectors corresponding to $\{\sigma_i(\mathbf{c})\}_{i=1}^n$ are continuous with respect to $\mathbf{c}$ around $\mathbf{c}^*$ (cf. [3]). Thus, in this case, one can certainly apply Newton's method for solving the nonlinear equation $\mathbf{f}(\mathbf{c}) = \mathbf{0}$ to produce Newton's method for solving ISVP (1.2). However, Newton's method for the ISVP (1.2) requires solving a complete singular value problem for the matrix $A(\mathbf{c})$ at each outer iteration. This sometimes makes it inefficient from the viewpoint of practical calculations especially when the problem size is large. Chu designed in [5] a Newton-type method for solving the ISVP (1.2) which requires computing approximate singular vectors instead of singular vectors at each iteration. Under the assumption that the given singular values $\{\sigma_i^*\}_{i=1}^n$ are distinct and positive, the Newton-type method was proved in [2] to be quadratically convergent (in the root-convergence sense). To alleviate the over-solving problem, Bai *et al* designed in [3] an inexact version of the Newton-type method for the distinct and positive case where the approximate Jacobian equation was solved inexactly by adopting a suitable stopping criteria. Also under the assumption that the given singular values are distinct and positive, a convergence analysis for the inexact Newton-type method was presented in that paper and the superlinear convergence was proved. Recently, the inexact Newton-type method (and so the Newton-type method) was extended in [24] to the multiple but positive case:

$$\sigma_1^* = \cdots = \sigma_s^* > \sigma_{s+1}^* > \cdots > \sigma_n^* > 0, \tag{1.5}$$

and a superlinear convergence result was established there without the distinction assumption of the given singular values. On the other hand, motivated by the Ulm-like method introduced in [23] for solving the IEP, Vong, Bai, and Jin presented in [28] a Ulm-like method for the ISVP (1.2). As noted in [28], the Ulm-like method avoids solving the (approximate) Jacobian equations and hence can reduce the unstability problem caused by the possible ill-conditioning in solving the (approximate) Jacobian equations. Moreover, the parallel computation techniques can be applied in the Ulm-like method to improve the computational efficiency. Again under the assumption (1.4), they showed that the proposed method converged at least quadratically. However, they comment that an interesting topic is to extend the Ulm-like method to the cases of multiple and zero singular values, which needs further investigation.

As noted in [5], zero singular values indicate rank deficiency and, to find a lower rank matrix in the generic affine subspace:

$$\mathcal{A}(\mathbf{c}) := \{A(\mathbf{c}) | \mathbf{c} \in \mathbb{R}^n\}$$

is intuitively a more difficult problem. Therefore, to our knowledge, the Ulm-like method has not been extended to the cases of multiple and zero singular values. In fact, few numerical method for solving the ISVP (1.2) (and so the square inverse singular value problems) with zero singular values has been explored. Furthermore, except the work in [24], there is also hardly any work on the numerical methods for solving the ISVP (1.2) with the multiple and positive case.

Motivated by the comment proposed in [28] (mentioned above), we study in the present paper the Ulm-like method for solving the square ISVP (i.e., the ISVP (1.2) with $m = n$) in the case when multiple and zero singular values present. The condition $\sigma_n^* > 0$ is removed and the appearance of multiple singular values is allowed here. That is, without loss of generality, $\{\sigma_i^*\}_{i=1}^n$ satisfies that

$$\sigma_1^* = \cdots = \sigma_s^* > \sigma_{s+1}^* \cdots > \sigma_{n-t}^* > \sigma_{n-t+1}^* = \ldots = \sigma_n^* = 0. \tag{1.6}$$

By modifying the Ulm-like method in [28] for the distinct and positive case, a Ulm-like method for solving the square ISVP with possible multiple and zero singular values is proposed. Under the nonsingularity assumption used by Shen *et al* in [24], we show that the proposed method converges quadratically (in the root sense) even when multiple and zero singular values are presented. It should be noted that the techniques used here for the convergence analysis are different from the ones for the distinct and positive case because of the absence of the differentiability of **f** and the continuity of the singular vectors as we explained above. Moreover, due to the occurrence of zero singular values, the techniques used in [24] for the Cayley transform method with multiple and positive singular values cannot be applied here. Finally, to illustrate our theoretical results, some numerical experiments are presented.

## 2 Preliminaries

Let $\mathbf{B}(\mathbf{x}, \delta)$ be the open ball in $\mathbb{R}^p$ with center $\mathbf{x} \in \mathbb{R}^p$ and radius $\delta > 0$. Let $\mathcal{O}(p)$ denote the set of all orthogonal matrices in $\mathbb{R}^{p\times p}$ and $I$ denote an identity matrix. Let $\|\cdot\|$ be the Euclidean vector norm or its induced matrix norm, and let $\|\cdot\|_F$ denote the Frobenius norm. Then,

$$\|M\| \le \|M\|_F \le \sqrt{q}\,\|M\|, \quad \text{for each } M \in \mathbb{R}^{p\times q}. \tag{2.1}$$

For any matrix $M \in \mathbb{R}^{p\times q}$, we use $M^{[l]_U}$ and $M^{[l]_L}$ to denote respectively the $l \times l$ upper left and lower right blocks of the matrix $M$. The symbol $\mathrm{Diag}(a_1, \ldots, a_n)$ denotes a diagonal matrix with $a_1$, $\ldots$, $a_n$ being its diagonal elements and $\mathrm{diag}(M) := (m_{11}, \ldots, m_{nn})^T$ denotes a vector containing the diagonal elements of an $n \times n$ matrix $M := (m_{ij})$. Let $\{\sigma_i^*\}_{i=1}^n$ be the given singular values satisfying (1.6). Write

$$\boldsymbol{\sigma}^* := (\sigma_1^*, \ldots, \sigma_n^*)^T \quad \text{and} \quad \Sigma^* := \mathrm{Diag}(\sigma_1^*, \ldots, \sigma_n^*) \in \mathbb{R}^{n\times n}. \tag{2.2}$$

Let $\{A_i\}_{i=0}^n \subset \mathbb{R}^{n\times n}$. Let $\mathbf{c} \in \mathbb{R}^n$ and $A(\mathbf{c})$ be defined by (1.1). Let $\{\sigma_i(\mathbf{c})\}_{i=1}^n$ stand for the singular values of $A(\mathbf{c})$ with the order $\sigma_1(\mathbf{c}) \geq \sigma_2(\mathbf{c}) \geq \cdots \geq \sigma_n(\mathbf{c}) \geq 0$. Write

$$\Sigma(\mathbf{c}) := \mathrm{Diag}(\sigma_1(\mathbf{c}),\ \ldots,\ \sigma_n(\mathbf{c})) \in \mathbb{R}^{n\times n}.$$

Define

$$\mathcal{W}(\mathbf{c}) := \{[U(\mathbf{c}),\ V(\mathbf{c})] \mid U(\mathbf{c})^T A(\mathbf{c}) V(\mathbf{c}) = \Sigma(\mathbf{c}),\ U(\mathbf{c}),\ V(\mathbf{c}) \in \mathcal{O}(n)\}.$$

As in [28], we ignore the choice of possible sign for $[U(\mathbf{c}),\ V(\mathbf{c})]$. For each $[U(\mathbf{c}),\ V(\mathbf{c})] \in \mathcal{W}(\mathbf{c})$, we write $U(\mathbf{c}) := [U^{(1)}(\mathbf{c}),\ U^{(2)}(\mathbf{c}),\ U^{(3)}(\mathbf{c})]$ and $V(\mathbf{c}) := [V^{(1)}(\mathbf{c}),\ V^{(2)}(\mathbf{c}),\ V^{(3)}(\mathbf{c})]$, where $U^{(1)}(\mathbf{c})$, $V^{(1)}(\mathbf{c}) \in \mathbb{R}^{n\times s}$ and $U^{(3)}(\mathbf{c}),\ V^{(3)}(\mathbf{c}) \in \mathbb{R}^{n\times t}$. Throughout this paper, we suppose that $\mathbf{c}^*$ is a solution of the square ISVP. For $i = 1$ and $i = 3$, define

$$\Pi_{U,i} = U^{(i)}(\mathbf{c}^*)U^{(i)}(\mathbf{c}^*)^T \quad \text{and} \quad \Pi_{V,i} = V^{(i)}(\mathbf{c}^*)V^{(i)}(\mathbf{c}^*)^T. \tag{2.3}$$

We first present some auxiliary lemmas. In particular, Lemma 2.1 gives a perturbation bound for the inverse which is known in [11, pp.58–59]; Lemma 2.2 is a direct consequence of the Cholesky factorization (cf. [10, Lemma 3.1]); while Lemmas 2.3 and 2.4 have been presented respectively in [2, Lemma 2] and [22, Lemma 4.1].

**Lemma 2.1.** *Let $A,\ B \in \mathbb{R}^{p\times p}$. Assume that $B$ is nonsingular and $\|B^{-1}\| \cdot \|A - B\| < 1$. Then $A$ is nonsingular and moreover*

$$\|A^{-1}\| \leq \frac{\|B^{-1}\|}{1 - \|B^{-1}\| \cdot \|A - B\|}.$$

**Lemma 2.2.** *Let $M \in \mathbb{R}^{p\times q}$ where $p \geq q$. Let $W = (w_{ij})$ be a $q \times q$ nonsingular upper triangle matrix such that $w_{11} > 0$ and $W^T W = I - M^T M$. Then there exist two numbers $\epsilon \in (0,\ 1)$ and $\alpha \in (0, +\infty)$ such that the following implication holds:*

$$\|M\| \leq \epsilon \Longrightarrow \|I - W\| \leq \alpha \|M\|^2. \tag{2.4}$$

**Lemma 2.3.** *There exists a constant $\alpha \in (0, +\infty)$ such that for any $\mathbf{c},\ \bar{\mathbf{c}} \in \mathbb{R}^n$,*

$$\|A(\mathbf{c}) - A(\bar{\mathbf{c}})\| \leq \alpha \|\mathbf{c} - \bar{\mathbf{c}}\|.$$

**Lemma 2.4.** *Suppose that $\hat{A} \in \mathcal{S}(n)$. Then there exist positive constants $\beta$ and $\kappa$ such that*

$$\min_{\hat{Q}\in\mathcal{O}(n),\ \hat{Q}^T\hat{A}\hat{Q}\in\mathcal{D}(n)} \|Q - \hat{Q}\| \leq \beta\|A - \hat{A}\|, \text{ whenever } A \in \mathcal{S}(n),\ Q \in \mathcal{O}(n),\ Q^T A Q \in \mathcal{D}(n),\ \|A - \hat{A}\| \leq \kappa.$$

The following lemma is also needed, the proof of which is similar to that of [24, Lemma 2.4]. For the sake of completeness, we present the proof.

**Lemma 2.5.** *Let $Z \in \mathbb{R}^{n\times n}$. Suppose that the skew-symmetric matrices $H \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times n}$ satisfy*

$$H\Sigma^* - \Sigma^* K = Z. \tag{2.5}$$

*Then we have*

$$[H]_{ij} = \frac{[Z]_{ij}}{\sigma_j^*}, \quad n - t + 1 \leq i \leq n, \quad 1 \leq j \leq n - t, \tag{2.6}$$

$$[K]_{ij} = -\frac{[Z]_{ij}}{\sigma_i^*}, \quad 1 \leq i \leq n - t, \quad n - t + 1 \leq j \leq n, \tag{2.7}$$

$$[H]_{ij} = \frac{1}{(\sigma_j^*)^2 - (\sigma_i^*)^2}\left(\sigma_j^*[Z]_{ij} + \sigma_i^*[Z]_{ji}\right), \quad s+1 \le i \le n-t, \quad 1 \le j \le n-t, \quad i > j. \tag{2.8}$$

$$[K]_{ij} = \frac{1}{(\sigma_j^*)^2 - (\sigma_i^*)^2}\left(\sigma_j^*[Z]_{ji} + \sigma_i^*[Z]_{ij}\right), \quad s+1 \le i \le n-t, \quad 1 \le j \le n-t, \quad i > j. \tag{2.9}$$

*Proof.* Let $\Sigma^{(11)} := \mathrm{Diag}(\sigma_1^*, \ldots, \sigma_{n-t}^*)$. Then, by the definition of $\Sigma^{(11)}$, one sees that

$$\Sigma^* = \begin{bmatrix} \Sigma^{(11)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{0}$ is a zero matrix of appropriate size. Write

$$H := \begin{bmatrix} H^{(11)} & H^{(12)} \\ H^{(21)} & H^{(22)} \end{bmatrix}, \quad K := \begin{bmatrix} K^{(11)} & K^{(12)} \\ K^{(21)} & K^{(22)} \end{bmatrix}, \quad \text{and} \quad Z := \begin{bmatrix} Z^{(11)} & Z^{(12)} \\ Z^{(21)} & Z^{(22)} \end{bmatrix},$$

where $H^{(11)}, K^{(11)}, Z^{(11)} \in \mathbb{R}^{(n-t)\times(n-t)}$. Then it follows from (2.5) that

$$H^{(11)}\Sigma^{(11)} - \Sigma^{(11)}K^{(11)} = Z^{(11)}, \tag{2.10}$$

$$-\Sigma^{(11)}K^{(12)} = Z^{(12)}$$

and

$$H^{(21)}\Sigma^{(11)} = Z^{(21)}.$$

Thus, (2.6) and (2.7) are seen to hold. It remains to prove (2.8) and (2.9). Indeed, noting that $(H^{(11)})^T = -H^{(11)}$ and $(K^{(11)})^T = -K^{(11)}$, we have by (2.10) that

$$-\Sigma^{(11)}H^{(11)} + K^{(11)}\Sigma^{(11)} = (Z^{(11)})^T. \tag{2.11}$$

Eliminating the matrix $K^{(11)}$ in (2.10) and (2.11) gives rise to

$$H^{(11)}(\Sigma^{(11)})^2 - (\Sigma^{(11)})^2 H^{(11)} = Z^{(11)}\Sigma^{(11)} + \Sigma^{(11)}(Z^{(11)})^T.$$

Equating the off-diagonal elements yields

$$[H^{(11)}]_{ij} = \frac{1}{(\sigma_j^*)^2 - (\sigma_i^*)^2}\left(\sigma_j[Z^{(11)}]_{ij} + \sigma_i[Z^{(11)}]_{ji}\right), \quad s+1 \le i \le n-t, \quad 1 \le j \le n-t, \quad i > j.$$

Similarly, we can prove

$$[K^{(11)}]_{ij} = \frac{1}{(\sigma_j^*)^2 - (\sigma_i^*)^2}\left(\sigma_j[Z^{(11)}]_{ji} + \sigma_i[Z^{(11)}]_{ij}\right), \quad s+1 \le i \le n-t, \quad 1 \le j \le n-t, \quad i > j.$$

Consequently, (2.8) and (2.9) are shown; hence the proof is complete. □

The following lemma describes the continuity properties of the singular vectors. Recall that we ignore the choice of possible sign for $[U(\mathbf{c}), V(\mathbf{c})]$ where $U(\mathbf{c}) := [U^{(1)}(\mathbf{c}), U^{(2)}(\mathbf{c}), U^{(3)}(\mathbf{c})]$ and $V(\mathbf{c}) := [V^{(1)}(\mathbf{c}), V^{(2)}(\mathbf{c}), V^{(3)}(\mathbf{c})]$. Recall also that $\Pi_{U,i}$ and $\Pi_{V,i}$ are defined by (2.3) for $i = 1, 3$.

**Lemma 2.6.** *There exist two numbers $\delta \in (0, 1)$ and $\gamma \in [1, +\infty)$ such that for any $\mathbf{c} \in \mathbf{B}(\mathbf{c}^*, \delta)$ and $[U(\mathbf{c}), V(\mathbf{c})] \in \mathcal{W}(\mathbf{c})$ , the following assertions hold:*

(i) $\|U^{(2)}(\mathbf{c}) - U^{(2)}(\mathbf{c}^*)\| \le \gamma\|\mathbf{c} - \mathbf{c}^*\|$ *and* $\|(I - \Pi_{U,i})U^{(i)}(\mathbf{c})\| \le \gamma\|\mathbf{c} - \mathbf{c}^*\|$ *for* $i = 1, 3$*;*

(ii) $\|V^{(2)}(\mathbf{c}) - V^{(2)}(\mathbf{c}^*)\| \leq \gamma\|\mathbf{c} - \mathbf{c}^*\|$ *and* $\|(I - \Pi_{V,i})V^{(i)}(\mathbf{c})\| \leq \gamma\|\mathbf{c} - \mathbf{c}^*\|$ *for* $i = 1,\ 3$.

*Proof.* The proof of assertions (i) and (ii) are similar and so we only prove assertion (ii) for brevity. In fact, the estimate of $\|U^{(2)}(\mathbf{c}) - U^{(2)}(\mathbf{c}^*)\|$ is clear and can be found in [2, 28]. For the estimate of $\|(I - \Pi_{U,i})U^{(i)}(\mathbf{c})\|$, define

$$\mathcal{V}(\mathbf{c}) := \{V(\mathbf{c}) \in \mathcal{O}(n) \mid [U(\mathbf{c}),\ V(\mathbf{c})] \in \mathcal{W}(\mathbf{c}) \text{ for some } U(\mathbf{c}) \in \mathcal{O}(n)\}, \quad \text{for any } \mathbf{c} \in \mathbb{R}^n,$$

and

$$\tilde{\mathcal{V}}(\mathbf{c}) := \{V(\mathbf{c}) \in \mathcal{O}(n) \mid V(\mathbf{c})^T A(\mathbf{c})^T A(\mathbf{c}) V(\mathbf{c}) = \Sigma(\mathbf{c})^T \Sigma(\mathbf{c})\}, \quad \text{for any } \mathbf{c} \in \mathbb{R}^n.$$

Then $\mathcal{V}(\mathbf{c}) = \tilde{\mathcal{V}}(\mathbf{c})$ (see the arguments of [22, Proposition 2.4]). Let $\alpha$, $\beta$, and $\kappa$ be determined by Lemmas 2.3 and 2.4. Let $\delta$ satisfy

$$0 < \delta < \min\left\{1,\ \frac{\kappa}{\alpha^2 + 2\alpha\|A(\mathbf{c}^*)\|}\right\}. \tag{2.12}$$

Suppose that $\mathbf{c} \in \mathbf{B}(\mathbf{c}^*,\ \delta)$ and $[U(\mathbf{c}),\ V(\mathbf{c})] \in \mathcal{W}(\mathbf{c})$. Then, $V(\mathbf{c}) \in \mathcal{V}(\mathbf{c})$ and so $V(\mathbf{c}) \in \tilde{\mathcal{V}}(\mathbf{c})$. Note by Lemma 2.3 that

$$\|A(\mathbf{c}) - A(\mathbf{c}^*)\| \leq \alpha\|\mathbf{c} - \mathbf{c}^*\|. \tag{2.13}$$

Since $\|\mathbf{c} - \mathbf{c}^*\| \leq \delta < 1$ and

$$A(\mathbf{c})A(\mathbf{c})^T - A(\mathbf{c}^*)A(\mathbf{c}^*)^T = (A(\mathbf{c}) - A(\mathbf{c}^*))(A(\mathbf{c}) - A(\mathbf{c}^*))^T + A(\mathbf{c}^*)(A(\mathbf{c}) - A(\mathbf{c}^*))^T + (A(\mathbf{c}) - A(\mathbf{c}^*))A(\mathbf{c}^*)^T,$$

we deduce by (2.13) that

$$\|A(\mathbf{c})A(\mathbf{c})^T - A(\mathbf{c}^*)A(\mathbf{c}^*)^T\| \leq \alpha^2\|\mathbf{c} - \mathbf{c}^*\|^2 + 2\alpha\|A(\mathbf{c}^*)\| \cdot \|\mathbf{c} - \mathbf{c}^*\| \leq (\alpha^2 + 2\alpha\|A(\mathbf{c}^*)\|)\|\mathbf{c} - \mathbf{c}^*\|. \tag{2.14}$$

Thus, using (2.12) and the fact of $\|\mathbf{c} - \mathbf{c}^*\| \leq \delta$, we further derive that

$$\|A(\mathbf{c})A(\mathbf{c})^T - A(\mathbf{c}^*)A(\mathbf{c}^*)^T\| \leq (\alpha^2 + 2\alpha\|A(\mathbf{c}^*)\|)\delta \leq \kappa.$$

Hence, thanks to Lemma 2.4, there exists a $V(\mathbf{c}^*) \in \tilde{\mathcal{V}}(\mathbf{c}^*)$ (and so $V(\mathbf{c}^*) \in \mathcal{V}(\mathbf{c}^*)$) such that the following inequality holds:

$$\|V(\mathbf{c}) - V(\mathbf{c}^*)\| \leq \beta\|A(\mathbf{c})A(\mathbf{c})^T - A(\mathbf{c}^*)A(\mathbf{c}^*)^T\|$$

which together with (2.14) gives that

$$\|V(\mathbf{c}) - V(\mathbf{c}^*)\| \leq \beta(\alpha^2 + 2\alpha\|A(\mathbf{c}^*)\|)\|\mathbf{c} - \mathbf{c}^*\|. \tag{2.15}$$

By (2.3), one checks that $\|I - \Pi_{V,i}\| \leq 1$ and $(I - \Pi_{V,i})V^{(i)}(\mathbf{c}) = (I - \Pi_{V,i})[V^{(i)}(\mathbf{c}) - V^{(i)}(\mathbf{c}^*)]$. Then it follows from (2.15) that

$$\|(I - \Pi_{V,i})V^{(i)}(\mathbf{c})\| \leq \|V^{(i)}(\mathbf{c}) - V^{(i)}(\mathbf{c}^*)\| \leq \|V(\mathbf{c}) - V(\mathbf{c}^*)\| \leq \beta(\alpha^2 + 2\alpha\|A(\mathbf{c}^*)\|)\|\mathbf{c} - \mathbf{c}^*\|$$

Therefore, we conclude that there exist positive numbers $\delta \in (0,\ 1)$ and $\gamma \in [1, +\infty)$ such that for any $\mathbf{c} \in \mathbf{B}(\mathbf{c}^*,\ \delta)$ assertion (ii) hold. □

Suppose that $U := [U^{(1)},\ U^{(2)},\ U^{(3)}]$ and $V := [V^{(1)},\ V^{(2)},\ V^{(3)}] \in \mathcal{O}(n)$ where $U^{(1)},\ V^{(1)} \in \mathbb{R}^{n\times s}$ and $U^{(3)},\ U^{(3)} \in \mathbb{R}^{n\times t}$. Let us construct two orthogonal matrices of singular vectors of $A(\mathbf{c}^*)$,

$$\tilde{U}(\mathbf{c}^*) := [\tilde{U}^{(1)}(\mathbf{c}^*),\ U^{(2)}(\mathbf{c}^*),\ \tilde{U}^{(3)}(\mathbf{c}^*)] \quad \text{and} \quad \tilde{V}(\mathbf{c}^*) := [\tilde{V}^{(1)}(\mathbf{c}^*),\ V^{(2)}(\mathbf{c}^*),\ \tilde{V}^{(3)}(\mathbf{c}^*)],$$

which is, in some sense, close to $U$ and $V$ respectively. To find $\tilde{U}^{(1)}(\mathbf{c}^*)$ and $\tilde{V}^{(1)}(\mathbf{c}^*)$, we start by considering the matrix $\Pi_{U,1}U^{(1)}$ and $\Pi_{V,1}V^{(1)}$ whose columns are singular vectors for $\sigma_1(\mathbf{c}^*)$. Using the similar way, we can find $\tilde{U}^{(3)}(\mathbf{c}^*)$ and $\tilde{V}^{(3)}(\mathbf{c}^*)$. Then, for $i = 1$ and $i = 3$, we form the QR factorization of $\Pi_{U,i}U^{(i)}$ and $\Pi_{V,i}V^{(i)}$:

$$\Pi_{U,i}U^{(i)} = \tilde{U}^{(i)}(\mathbf{c}^*)R_U^{(i)}, \quad \Pi_{V,i}V^{(i)} = \tilde{V}^{(i)}(\mathbf{c}^*)R_V^{(i)}, \tag{2.16}$$

where $R_U^{(i)}$ $R_V^{(i)}$ are nonsingular upper triangular matrices, and $\tilde{U}^{(i)}(\mathbf{c}^*)$, $\tilde{V}^{(i)}(\mathbf{c}^*)$ are matrices whose columns are orthonormal. Clearly, the columns of $\tilde{U}(\mathbf{c}^*)$ and $\tilde{V}(\mathbf{c}^*)$ are singular vectors of $A(\mathbf{c}^*)$. That is, $[\tilde{U}(\mathbf{c}^*)\ \tilde{V}(\mathbf{c}^*)] \in \mathcal{W}(\mathbf{c}^*)$. Let the skew-symmetric matrices $\tilde{X}$ and $\tilde{Y}$ defined by

$$e^{\tilde{X}} = U^T\tilde{U}(\mathbf{c}^*) \quad \text{and} \quad e^{\tilde{Y}} = V^T\tilde{V}(\mathbf{c}^*). \tag{2.17}$$

Finally, let us define the error matrices $E_U$ and $E_V$:

$$E_U := [E_U^{(1)},\ E_U^{(2)},\ E_U^{(3)}] \quad \text{and} \quad E_V := [E_V^{(1)},\ E_V^{(2)},\ E_V^{(3)}], \tag{2.18}$$

where

$$E_U^{(i)} := (I - \Pi_{U,i})U^{(i)} \quad \text{and} \quad E_V^{(i)} := (I - \Pi_{V,i})V^{(i)}, \quad i = 1,\ 3, \tag{2.19}$$

and

$$E_U^{(2)} := U^{(2)} - U^{(2)}(\mathbf{c}^*) \quad \text{and} \quad E_V^{(2)} := V^{(2)} - V^{(2)}(\mathbf{c}^*). \tag{2.20}$$

The following lemma plays an important role in the convergence analysis of the Ulm-like method.

**Lemma 2.7.** *There exist two numbers $\delta \in (0,\ 1)$ and $\gamma \in [1, +\infty)$ such that*

(i) *for any matrix $U \in \mathcal{O}(n)$ with $\|E_U\| < \delta$ , the skew-symmetric matrix $\tilde{X}$ defined by* (2.17) *satisfies*

$$\|\tilde{X}\|_F \le \gamma\|E_U\|, \quad \|\tilde{X}^{[s]_U}\|_F \le \gamma\|E_U\|^2 \quad \textit{and} \quad \|\tilde{X}^{[t]_L}\|_F \le \gamma\|E_U\|^2$$

(ii) *for any matrix $V \in \mathcal{O}(n)$ with $\|E_V\| < \delta$, the skew-symmetric matrix $\tilde{Y}$ defined by* (2.17) *satisfies*

$$\|\tilde{Y}\|_F \le \gamma\|E_V\|, \quad \|\tilde{Y}^{[s]_U}\|_F \le \gamma\|E_V\|^2 \quad \textit{and} \quad \|\tilde{Y}^{[t]_L}\|_F \le \gamma\|E_V\|^2.$$

*Proof.* Since the proofs of assertions (i) and (ii) are similar, we only provide the proof of assertion (i). For this purpose, let $U \in \mathcal{O}(n)$. Note that

$$e^{\tilde{X}} = U^T\tilde{U}(\mathbf{c}^*) = \begin{pmatrix} (U^{(1)})^T\tilde{U}^{(1)}(\mathbf{c}^*) & (U^{(1)})^TU^{(2)}(\mathbf{c}^*) & (U^{(1)})^T\tilde{U}^{(3)}(\mathbf{c}^*) \\ (U^{(2)})^T\tilde{U}^{(1)}(\mathbf{c}^*) & (U^{(2)})^TU^{(2)}(\mathbf{c}^*) & (U^{(2)})^T\tilde{U}^{(3)}(\mathbf{c}^*) \\ (U^{(3)})^T\tilde{U}^{(1)}(\mathbf{c}^*) & (U^{(3)})^TU^{(2)}(\mathbf{c}^*) & (U^{(3)})^T\tilde{U}^{(3)}(\mathbf{c}^*) \end{pmatrix}.$$

Since $e^{\tilde{X}} = I + \tilde{X} + O(\|\tilde{X}\|^2)$, it suffices to prove

$$U^T\tilde{U}(\mathbf{c}^*) = \begin{pmatrix} (U^{(1)})^T\tilde{U}^{(1)}(\mathbf{c}^*) & (E_U^{(1)})^TU^{(2)}(\mathbf{c}^*) & (E_U^{(1)})^T\tilde{U}^{(3)}(\mathbf{c}^*) \\ (E_U^{(2)})^T\tilde{U}^{(1)}(\mathbf{c}^*) & I + (E_U^{(2)})^TU^{(2)}(\mathbf{c}^*) & (E_U^{(2)})^T\tilde{U}^{(3)}(\mathbf{c}^*) \\ (E_U^{(3)})^T\tilde{U}^{(1)}(\mathbf{c}^*) & (E_U^{(3)})^TU^{(2)}(\mathbf{c}^*) & (U^{(3)})^T\tilde{U}^{(3)}(\mathbf{c}^*) \end{pmatrix}.$$

and, to prove there exist $\delta > 0$ and $\gamma > 0$ such that, if $\|E_U\| \le \delta$, then

$$\max\{\|I - (U^{(1)})^T\tilde{U}^{(1)}(\mathbf{c}^*)\|,\ \|I - (U^{(3)})^T\tilde{U}^{(3)}(\mathbf{c}^*)\|\} \le \gamma\|E_U\|^2. \tag{2.21}$$

Note by (2.16), (2.19) and the orthogonality of the matrix $[\tilde{U}^{(1)}(\mathbf{c}^*),\ U^{(2)}(\mathbf{c}^*),\ \tilde{U}^{(3)}(\mathbf{c}^*)]$ that

$$(U^{(1)})^TU^{(2)}(\mathbf{c}^*) = (E_U^{(1)} + \tilde{U}^{(1)}(\mathbf{c}^*)R_U^{(1)})^TU^{(2)}(\mathbf{c}^*) = (E_U^{(1)})^TU^{(2)}(\mathbf{c}^*). \tag{2.22}$$

Similarly, one can prove that

$$(U^{(1)})^T\tilde{U}^{(3)}(\mathbf{c}^*) = (E_U^{(1)})^T\tilde{U}^{(3)}(\mathbf{c}^*), \tag{2.23}$$

$$(U^{(3)})^T\tilde{U}^{(1)}(\mathbf{c}^*) = (E_U^{(3)})^T\tilde{U}^{(1)}(\mathbf{c}^*), \tag{2.24}$$

and

$$(U^{(3)})^TU^{(2)}(\mathbf{c}^*) = (E_U^{(3)})^TU^{(2)}(\mathbf{c}^*). \tag{2.25}$$

Using (2.20) and the orthogonality of the matrix $[\tilde{U}^{(1)}(\mathbf{c}^*)\ U^{(2)}(\mathbf{c}^*)\ \tilde{U}^{(3)}(\mathbf{c}^*)]$, we get that

$$(U^{(2)})^T\tilde{U}^{(1)}(\mathbf{c}^*) = (E_U^{(2)} + U^{(2)}(\mathbf{c}^*))^T\tilde{U}^{(1)}(\mathbf{c}^*) = (E_U^{(2)})^T\tilde{U}^{(1)}(\mathbf{c}^*). \tag{2.26}$$

Similarly, we can also prove that

$$(U^{(2)})^TU^{(2)}(\mathbf{c}^*) = I + (E_U^{(2)})^TU^{(2)}(\mathbf{c}^*) \quad \text{and} \quad (U^{(2)})^T\tilde{U}^{(3)}(\mathbf{c}^*) = (E_U^{(2)})^T\tilde{U}^{(3)}(\mathbf{c}^*).$$

It remains to prove that there exist $\delta > 0$ and $\gamma > 0$ such that

$$\|E_U\| \le \delta \Longrightarrow \max\{\|I - (U^{(1)})^T\tilde{U}^{(1)}(\mathbf{c}^*)\|,\ \|I - (U^{(3)})^T\tilde{U}^{(3)}(\mathbf{c}^*)\|\} \le \gamma\|E_U\|^2. \tag{2.27}$$

Note by the definition of $E_U^{(1)}$ that $E_U^{(1)} = (I - \Pi_{U,1})U^{(1)}$. Then, thanks to the fact $\Pi_{U,1}^T\Pi_{U,1} = \Pi_{U,1}$, one has that $(E_U^{(1)})^T\Pi_{U,1}U^{(1)} = \mathbf{0}$ which gives

$$(E_U^{(1)})^TU^{(1)} = (E_U^{(1)})^T(E_U^{(1)} + \Pi_{U,1}U^{(1)}) = (E_U^{(1)})^TE_U^{(1)}.$$

Since $(U^{(1)})^TU^{(1)} = I$, one can easily check that

$$(\Pi_{U,1}U^{(1)})^T(\Pi_{U,1}U^{(1)}) = (U^{(1)} - E_U^{(1)})^T(U^{(1)} - E_U^{(1)}) = I - (E_U^{(1)})^TE_U^{(1)}. \tag{2.28}$$

Noting that the columns of $\tilde{U}^{(1)}(\mathbf{c}^*)$ are orthonormal, we get from (2.16) that

$$(R_U^{(1)})^TR_U^{(1)} = (\Pi_{U,1}U^{(1)})^T\Pi_{U,1}U^{(1)}.$$

This, together with (2.28), implies that

$$(R_U^{(1)})^TR_U^{(1)} = I - (E_U^{(1)})^TE_U^{(1)}. \tag{2.29}$$

To proceed, we apply Lemma 2.2 to choose $\epsilon_1 \in (0,\ 1)$ and $\alpha_1 \in (0, +\infty)$ such that implication (2.4) holds with $\epsilon_1, \alpha_1$ in place of $\epsilon, \alpha$. In particular, applying (2.4) to $\{R_U^{(1)}, E_U^{(1)}\}$ in place of $\{W, M\}$, we have that

$$\|E_U^{(1)}\| \le \epsilon_1 \Longrightarrow \|I - R_U^{(1)}\| \le \alpha_1\|E_U^{(1)}\|^2 \tag{2.30}$$

thanks to (2.29) (noting that the first element of $R_U^{(1)}$ is positive). Since $\|E_U^{(1)}\| \leq \|E_U\|$ by (2.18), it follows from (2.30) that

$$\|E_U\| \leq \epsilon_1 \Longrightarrow \|I - R_U^{(1)}\| \leq \alpha_1 \|E_U\|^2. \tag{2.31}$$

Furthermore, noting that $\Pi_{U,1}\tilde{U}^{(1)}(\mathbf{c}^*) = \tilde{U}^{(1)}(\mathbf{c}^*)$ and $\Pi_{U,1}^T = \Pi_{U,1}$, we have by (2.16) that

$$R_U^{(1)} = (\tilde{U}^{(1)}(\mathbf{c}^*))^T \Pi_{U,1} U^{(1)} = (\tilde{U}^{(1)}(\mathbf{c}^*))^T U^{(1)}.$$

Substituting this into (2.31), one has that

$$\|E_U\| \leq \epsilon_1 \Longrightarrow \|I - (\tilde{U}^{(1)}(\mathbf{c}^*))^T U^{(1)}\| \leq \alpha_1 \|E_U\|^2, \tag{2.32}$$

With the similar arguments for proving (2.32), we can prove that there exist $\epsilon_1 > 0$ and $\alpha_1 > 0$ such that

$$\|E_U\| \leq \epsilon_1 \Longrightarrow \|I - (\tilde{U}^{(3)}(\mathbf{c}^*))^T U^{(3)}\| \leq \alpha_1 \|E_U\|^2. \tag{2.33}$$

This together with (2.32) implies that (2.27) holds for $0 < \delta \leq \epsilon_1$ and $\gamma \geq \max\{1,\ \alpha_1\}$. Therefore, the proof is complete. □

Now we present the definitions and some properties of the B-differential Jacobian, the generalized Jacobian and the relative generalized Jacobian. For this, let $\mathbf{g} : \mathbb{R}^p \to \mathbb{R}^q$ be a locally Lipschitz continuous function. Let $\mathbf{g}'$ be the Fréchet derivative of $\mathbf{g}$ whenever it exists and $D_{\mathbf{g}}$ be the set of differentiable points of $\mathbf{g}$. Recall from [8, 20] that the B-differential Jacobian of $\mathbf{g}$ at $\mathbf{x} \in \mathbb{R}^p$ is defined by

$$\partial_B \mathbf{g}(\mathbf{x}) := \{J \in \mathbb{R}^{q\times p} \mid J = \lim_{\mathbf{x}_k \to \mathbf{x}} \mathbf{g}'(\mathbf{x}_k),\ \mathbf{x}_k \in D_{\mathbf{g}}\}.$$

Consider the composite nonsmooth function:

$$\mathbf{g} := \varphi \circ \psi, \tag{2.34}$$

where $\varphi : \mathbb{R}^l \to \mathbb{R}^q$ is nonsmooth but of special structure and $\psi : \mathbb{R}^p \to \mathbb{R}^l$ is continuously differentiable. Let $\mathcal{S}$ be a subset of $\mathbb{R}^n$ and cl$\mathcal{S}$ denote the closure of $\mathcal{S}$. The generalized Jacobian $\partial_Q \mathbf{g}(\cdot)$ and relative generalized Jacobian $\partial_{Q|\mathcal{S}} \mathbf{g}(\cdot)$ at $\mathbf{x} \in \mathbb{R}^p$, which were introduced respectively in [18] and [25], are defined as follows:

$$\partial_Q \mathbf{g}(\mathbf{x}) := \partial_B(\varphi(\psi(\mathbf{x})))\psi'(\mathbf{x});$$

$$\partial_{Q|\mathcal{S}} \mathbf{g}(\mathbf{x}) := \{J \mid J \text{ is a limit of } G_k \in \partial_Q \mathbf{g}(\mathbf{y}_k),\ \mathbf{y}_k \in \mathcal{S},\ \mathbf{y}_k \to \mathbf{x}\}.$$

The following lemma is known in [24, Proposition 2.1].

**Lemma 2.8.** *Let $\bar{\mathbf{x}} \in \mathbb{R}^p$ and let $\mathcal{S}$ be a subset of $\mathbb{R}^p$. Let $g$ be defined by* (2.34)*. Then $\partial_B \mathbf{g}(\bar{\mathbf{x}})$ and $\partial_Q \mathbf{g}(\bar{\mathbf{x}})$ are nonempty and compact, and so is $\partial_{Q|\mathcal{S}} \mathbf{g}(\bar{\mathbf{x}})$ if $\bar{\mathbf{x}} \in$ cl$\mathcal{S}$.*

In the remainder of the present paper, let

$$S := \{\mathbf{c} \in \mathbb{R}^n \mid A(\mathbf{c}) \text{ has positive and distinct singular values}\}.$$

For any matrix $M \in \mathbb{R}^{n\times n}$, we use $\{\sigma_i(M)\}_{i=1}^n$ to denote the eigenvalues of $M$ with $\sigma_1(M) \geq \ldots \geq \sigma_n(M) \geq 0$. Define the operator $\boldsymbol{\sigma} : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^n$ by

$$\boldsymbol{\sigma}(M) := (\sigma_1(M), \ldots, \sigma_n(M))^T, \quad \text{for any} \quad M \in \mathbb{R}^{n\times n}.$$

Recall that the operators $A$ and $\mathbf{f}$ are defined by (1.1) and (1.3) respectively. Then

$$\mathbf{f} = \boldsymbol{\sigma} \circ A - \boldsymbol{\sigma}^*.$$

Thus we have the following two lemmas. In particular, Lemma 2.9, which has been proved in [24], gives the B-differential Jacobian, generalized Jacobian, and relative generalized Jacobian of $\mathbf{f}$ at $\mathbf{c}$. While Lemma 2.10, which is a direct consequence of [22, Lemma 2.1], gives a perturbation bound for the inverses of B-differential Jacobian, the generalized Jacobian and the relative generalized Jacobian.

**Lemma 2.9.** *Let* $\mathbf{f}$ *be defined by* (1.3). *Then we have the following assertions:*

(i) *If* $\mathbf{c} \in \mathbb{R}^n$ *such that* $\sigma_n(\mathbf{c}) > 0$, *then* $\partial_Q \mathbf{f}(\mathbf{c}) = \{J \mid [J]_{ij} = \mathbf{u}_i(\mathbf{c})^T A_j \mathbf{v}_i(\mathbf{c}),\ [U(\mathbf{c}),\ V(\mathbf{c})] \in \mathcal{W}(\mathbf{c})\}$.

(ii) *If* $\mathbf{c} \in S$, *then* $\mathbf{f}$ *is continuously differentiable at* $\mathbf{c}$ *and moreover* $\partial_B \mathbf{f}(\mathbf{c}) = \partial_Q \mathbf{f}(\mathbf{c}) = \{\mathbf{f}'(\mathbf{c})\}$;

(iii) $\partial_{Q|S} \mathbf{f}(\mathbf{c}) = \{J \mid J = \lim_{k \to +\infty} f'(\mathbf{y}^k)$ *with* $\{\mathbf{y}^k\} \subset S$ *and* $\mathbf{y}^k \to \mathbf{c}\}$.

**Lemma 2.10.** *Let* $\mathbf{c}^* \in \mathrm{cl}S$ *such that the matrix* $A(\mathbf{c}^*)$ *has singular values given by* (1.6). *Suppose that each* $J \in \partial_{Q|S}\mathbf{f}(\mathbf{c}^*)$ (*resp. each* $J \in \partial_B \mathbf{f}(\mathbf{c}^*)$, *each* $J \in \partial_Q \mathbf{f}(\mathbf{c}^*)$) *is nonsingular. Then there exist two numbers* $\delta \in (0,\ 1)$ *and* $\gamma \in [1,\ +\infty)$ *such that for any* $\mathbf{c} \in \mathbf{B}(\mathbf{c}^*, \delta)$,

$$\sup_{J \in \partial_{Q|S}\mathbf{f}(\mathbf{c})} \|J^{-1}\| (resp. \sup_{J \in \partial_B \mathbf{f}(\mathbf{c})} \|J^{-1}\|, \sup_{J \in \partial_Q \mathbf{f}(\mathbf{c})} \|J^{-1}\| \Big) \le \gamma, \tag{2.35}$$

*where we adopt the convention that* $\sup \varnothing = -\infty$.

# 3 The Ulm-like method and convergence analysis

In this section, we begin with the Ulm-like method for solving the square ISVP with the singular values given by (1.6). For the original idea of the Ulm-like method, one may refer to [28]. Compared with the Newton-type methods for solving the ISVP (1.2), the advantage of the Ulm-like method is that approximate Jacobian equations are not required to solve in each step. Clearly, in the case when $s = 1$ and $t = 0$, the method presented below is reduced to the Ulm-like method proposed in [28] (with $m = n$) for the positive and distinct case.

**The Ulm-like method**

1. Given $\mathbf{c}^0 \in \mathbb{R}^n$ and $B_0 \in \mathbb{R}^{n\times n}$. Compute the singular values $\{\sigma_i(\mathbf{c}^0)\}_{i=1}^n$, the orthonormal left singular vectors $\{\mathbf{u}_i(\mathbf{c}^0)\}_{i=1}^m$ and right singular vectors $\{\mathbf{v}_i(\mathbf{c}^0)\}_{i=1}^n$ of $A(\mathbf{c}^0)$. Write

$$U_0 := [\mathbf{u}_1^0, \ldots, \mathbf{u}_m^0] = [\mathbf{u}_1(\mathbf{c}^0), \ldots, \mathbf{u}_m(\mathbf{c}^0)],$$

$$V_0 := [\mathbf{v}_1^0, \ldots, \mathbf{v}_n^0] = [\mathbf{v}_1(\mathbf{c}^0), \ldots, \mathbf{v}_n(\mathbf{c}^0)].$$

Form the Jacobian matrix $J_0$ and the vector $\mathbf{b}^0$:

$$[J_0]_{ij} := (\mathbf{u}_i^0)^T A_j \mathbf{v}_i^0, \quad [\mathbf{b}^0]_i := (\mathbf{u}_i^0)^T A_0 \mathbf{v}_i^0, \quad 1 \le i,\ j \le n.$$

2. For $k = 0,\ 1,\ 2, \ldots$ until convergence, do:

(a) Compute the vector $\mathbf{c}^{k+1}$ by

$$\mathbf{c}^{k+1} := \mathbf{c}^k - B_k(J_k\mathbf{c}^k + \mathbf{b}^k - \boldsymbol{\sigma}^*). \tag{3.1}$$

(b) Form the matrix $W_k := U_k^T A(\mathbf{c}^{k+1})V_k$.

(c) Calculate the skew-symmetric matrices $X_k$ and $Y_k$:

$$[X_k]_{ij} := 0, \quad 1 \le i,\ j \le s \quad \text{or} \quad n-t+1 \le i,\ j \le n,$$

$$[X_k]_{ij} := -[X_k]_{ji} = \frac{[W_k]_{ij}}{\sigma_j^*}, \quad n-t+1 \le i \le n,\ 1 \le j \le n-t,$$

$$[X_k]_{ij} := -[X_k]_{ji} = \frac{\sigma_i^*[W_k]_{ji} + \sigma_j^*[W_k]_{ij}}{(\sigma_j^*)^2 - (\sigma_i^*)^2}, \quad s+1 \le i \le n-t,\ 1 \le j \le n-t,\ i > j,$$

$$[Y_k]_{ij} := 0, \quad n-t+1 \le i,\ j \le n,$$

$$[Y_k]_{ij} := -[Y_k]_{ji} = -\frac{[W_k]_{ij}}{\sigma_i^*}, \quad 1 \le i,\ j \le s,\ i > j,$$

$$[Y_k]_{ij} := -[Y_k]_{ji} = -\frac{[W_k]_{ij}}{\sigma_i^*}, \quad 1 \le i \le n-t, \quad n-t+1 \le j \le n,$$

$$[Y_k]_{ij} := -[Y_k]_{ji} = \frac{\sigma_i^*[W_k]_{ij} + \sigma_j^*[W_k]_{ji}}{(\sigma_j^*)^2 - (\sigma_i^*)^2}, \quad s+1 \le i \le n-t,\ 1 \le j \le n-t,\ i > j.$$

(d) Compute $U_{k+1} := [\mathbf{u}_1^{k+1}, \ldots, \mathbf{u}_m^{k+1}]$ and $V_{k+1} := [\mathbf{v}_1^{k+1}, \ldots, \mathbf{v}_n^{k+1}]$ by solving

$$\left(I + \frac{1}{2}X_k\right) U_{k+1}^T = \left(I - \frac{1}{2}X_k\right) U_k^T \tag{3.2}$$

and

$$\left(I + \frac{1}{2}Y_k\right) V_{k+1}^T = \left(I - \frac{1}{2}Y_k\right) V_k^T. \tag{3.3}$$

(e) Form the approximate Jacobian matrix $J_{k+1}$ and the vector $\mathbf{b}^{k+1}$:

$$[J_{k+1}]_{ij} := (\mathbf{u}_i^{k+1})^T A_j \mathbf{v}_i^{k+1}, \quad 1 \le i,\ j \le n; \tag{3.4}$$

$$[\mathbf{b}^{k+1}]_i := (\mathbf{u}_i^{k+1})^T A_0 \mathbf{v}_i^{k+1}, \quad 1 \le i \le n. \tag{3.5}$$

(f) Compute the matrix $B_{k+1}$:

$$B_{k+1} := 2B_k - B_k J_{k+1} B_k.$$

Now we present a convergence analysis for the Ulm-like method. Recall that we have assumed that the given singular values satisfy (1.6). There is no difficulty in generalizing all our results to an arbitrary set of given positive singular values. For the remainder of the present paper, let $\{\mathbf{c}^k\}$, $\{U_k\}$, $\{V_k\}$, $\{X_k\}$, $\{Y_k\}$, $\{B_k\}$, and $\{J_k\}$ be generated by the Ulm-like method with initial point $\mathbf{c}^0$. Let $\{\tilde{X}_k\}$, $\{\tilde{Y}_k\}$ be the skew-symmetric matrices defined by (2.16)–(2.17) with $\{U = U_k\}$ and $\{V = V_k\}$ respectively. Let $E_{U_k}$, $E_{V_k}$ be defined by (2.18)-(2.20) with $\{U = U_k\}$ and $\{V = V_k\}$ respectively. Then we have the following lemma.

**Lemma 3.1.** *There exist two numbers $\delta \in (0,\ 1)$ and $\gamma \in [1,\ +\infty)$ such that for any $k \geq 0$ and $[U(\mathbf{c}^*)\ V(\mathbf{c}^*)] \in \mathcal{W}(\mathbf{c}^*)$ with $\max\{\|E_{U_k}\|,\ \|E_{V_k}\|\} < \delta$, the following assertions hold:*

(i) $\|\Sigma^* + \tilde{X}_k\Sigma^* - \Sigma^*\tilde{Y}_k - U_k^T A(\mathbf{c}^*)V_k\| \leq \gamma(\|E_{U_k}\|^2 + \|E_{V_k}\|^2)$;

(ii) $\max\{\|U_{k+1} - U_k\|,\ \|V_{k+1} - V_k\|\} \leq \gamma(\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|E_{U_k}\| + \|E_{V_k}\|)$, *if* $\mathbf{c}^{k+1} \in \mathbf{B}(\mathbf{c}^*, \delta)$;

(iii) $\max\{\|E_{U_{k+1}}\|,\ \|E_{V_{k+1}}\|\} \leq \gamma[\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + (\|E_{U_k}\| + \|E_{V_k}\|)^2]$, *if* $\mathbf{c}^{k+1} \in \mathbf{B}(\mathbf{c}^*, \delta)$.

*Proof.* By Lemma 2.7, let $\delta_1 \in (0,\ 1)$ and $\gamma_1 \in [1,\ +\infty)$ be such that for any $k \geq 0$,

$$\|\tilde{X}_k\|_F \leq \gamma_1\|E_{U_k}\|, \quad \|\tilde{X}_k^{[s]_U}\|_F \leq \gamma_1\|E_{U_k}\|^2, \quad \|\tilde{X}_k^{[t]_L}\|_F \leq \gamma_1\|E_{U_k}\|^2, \tag{3.6}$$

$$\|\tilde{Y}_k\|_F \leq \gamma_1\|E_{V_k}\|, \quad \|\tilde{Y}_k^{[s]_U}\|_F \leq \gamma_1\|E_{V_k}\|^2, \quad \|\tilde{Y}_k^{[s]_L}\|_F \leq \gamma_1\|E_{V_k}\|^2, \tag{3.7}$$

when $\max\{\|E_{U_k}\|,\ \|E_{V_k}\} < \delta_1$. Let $\alpha$ the positive number determined in Lemma 2.3. Write

$$\eta_1 := (n^2 - s^2 - t^2) \max_{s \leq i < n-t} \left\{ \frac{1}{\sigma_{i+1}^* - \sigma_i^*},\ \frac{1}{\sigma_i^*} \right\}, \quad \eta_2 := \max\left\{2\eta_1\alpha,\ 4 + 2\gamma_1 + 8\gamma_1\eta_1\|\boldsymbol{\sigma}^*\|\right\}. \tag{3.8}$$

Set

$$\gamma := \max\left\{4\gamma_1^2\|\boldsymbol{\sigma}^*\|,\ \frac{9}{2}\sqrt{n}\gamma_1\eta_2\right\} \quad \text{and} \quad \delta := \min\left\{\delta_1,\ \frac{2}{3\gamma}\right\}. \tag{3.9}$$

Clearly, $\delta \in (0,\ 1)$ and $\gamma \in [1,\ +\infty)$. Below we prove that $\delta$ and $\gamma$ are as desired. For this purpose, we assume that $[U(\mathbf{c}^*),\ U(\mathbf{c}^*)] \in \mathcal{W}(\mathbf{c}^*)$. Let $k \geq 0$ be such that $\max\{\|E_{U_k}\|,\ \|E_{V_k}\|\} < \delta$. Then one has by (3.6), (3.7), and (3.9) that

$$\|\tilde{X}_k\|_F \leq \gamma_1\|E_{U_k}\| < \gamma\delta < 1 \quad \text{and} \quad \|\tilde{Y}_k\|_F \leq \gamma_1\|E_{U_k}\| < \gamma\delta < 1.$$

Thus, by direct computations, we have

$$\left\|\sum_{m=2}^{\infty} \frac{\tilde{X}_k^{m-2}}{m!}\right\|_F < \sum_{m=2}^{\infty} \frac{1}{m!} \leq \sum_{m=2}^{\infty} \frac{1}{m(m-1)} = 1. \tag{3.10}$$

Similarly,

$$\left\|\sum_{m=2}^{\infty} \frac{(-\tilde{Y}_k)^{m-2}}{m!}\right\|_F < 1, \qquad \left\|\sum_{m=1}^{\infty} \frac{(-\tilde{Y}_k)^{m-1}}{m!}\right\|_F < 2, \qquad \left\|\sum_{m=0}^{\infty} \frac{(-\tilde{Y}_k)^{m}}{m!}\right\|_F < 3. \tag{3.11}$$

Write

$$R_k := -\tilde{X}_k^2\Big(\sum_{m=2}^{\infty} \frac{\tilde{X}_k^{m-2}}{m!}\Big)\Sigma^*\Big(\sum_{m=0}^{\infty} \frac{(-\tilde{Y}_k)^m}{m!}\Big) - \Sigma^*\tilde{Y}_k^2 \sum_{m=2}^{\infty} \frac{(-\tilde{Y}_k)^{m-2}}{m!} + \tilde{X}_k\Sigma^*\tilde{Y}_k \sum_{m=1}^{\infty} \frac{(-\tilde{Y}_k)^{m-1}}{m!}. \tag{3.12}$$

Hence, one has by (3.10)–(3.12) that

$$\|R_k\|_F \leq (3\|\tilde{X}_k\|_F^2 + \|\tilde{Y}_k\|_F^2 + 2\|\tilde{X}_k\|_F \cdot \|\tilde{Y}_k\|_F) \cdot \|\Sigma^*\|_F \leq 4(\|\tilde{X}_k\|_F^2 + \|\tilde{Y}_k\|_F^2) \cdot \|\Sigma^*\|_F. \tag{3.13}$$

It follows from (2.2), (3.6), (3.7), and (3.13) that

$$\|R_k\|_F \leq 4\gamma_1^2\|\boldsymbol{\sigma}^*\|(\|E_{U_k}\|^2 + \|E_{V_k}\|^2). \tag{3.14}$$

On the other hand, noting that $e^{\tilde{X}_k} := U_k^T \tilde{U}(\mathbf{c}^*)$, $e^{\tilde{Y}_k} := V_k^T \tilde{V}(\mathbf{c}^*)$, and $[\tilde{U}(\mathbf{c}^*),\ \tilde{V}(\mathbf{c}^*)] \in \mathcal{W}(\mathbf{c}^*)$, we derive

$$e^{\tilde{X}_k}\Sigma^* e^{-\tilde{Y}_k} = U_k^T A(\mathbf{c}^*) V_k. \tag{3.15}$$

Thus, by (3.12) and the fact of $e^X = \sum\limits_{m=0}^{\infty} \frac{X^m}{m!}$, we can write (3.15) into the form

$$\Sigma^* + \tilde{X}_k\Sigma^* - \Sigma^*\tilde{Y}_k = U_k^T A(\mathbf{c}^*) V_k + R_k. \tag{3.16}$$

This together with (2.1), (3.14) as well as the definition of $\gamma$ implies that assertion (i) holds.

For the proof of assertions (ii) and (iii), we assume further that $\mathbf{c}^{k+1} \in \mathbf{B}(\mathbf{c}^*, \delta)$ (and so $\|\mathbf{c}^{k+1} - \mathbf{c}^*\| < \delta$). The estimates of $\|\tilde{X}_k - X_k\|$, $\|X_k\|$, and $\left\|\left(I - \frac{1}{2}X_k\right)^{-1}\right\|$ are needed first. Indeed, using (3.16) and applying Lemma 2.5 (to $\tilde{X}_k$, $\tilde{Y}_k$, $U_k^T A(\mathbf{c}^*)V_k + R_k - \Sigma^*$ in place of $H$, $K$ and $Z$), one has that

$$[\tilde{X}_k]_{ij} = \frac{(\mathbf{u}_i^k)^T A(\mathbf{c}^*)\mathbf{v}_j^k + [R_k]_{ij}}{\sigma_j^*}, \quad n-t+1 \le i \le n,\ 1 \le j \le n-t$$

and

$$[\tilde{X}_k]_{ij} = \frac{\sigma_j^*[(\mathbf{u}_i^k)^T A(\mathbf{c}^*)\mathbf{v}_j^k + [R_k]_{ij}] + \sigma_i^*[(\mathbf{u}_j^k)^T A(\mathbf{c}^*)\mathbf{v}_i^k + [R_k]_{ji}]}{(\sigma_j^*)^2 - (\sigma_i^*)^2},\ s+1 \le i \le n-t,\ 1 \le j \le n-t, i > j.$$

This together with the formulation of $X_k$ in the Ulm-like method yields that

$$[\tilde{X}_k]_{ij} - [X_k]_{ij} = \frac{(\mathbf{u}_i^k)^T \Delta_{k+1}\mathbf{v}_j^k + [R_k]_{ij}}{\sigma_j^*}, \quad n-t+1 \le i \le n,\ 1 \le j \le n-t, \tag{3.17}$$

and

$$\begin{aligned} & [\tilde{X}_k]_{ij} - [X_k]_{ij} \\ = & \frac{\sigma_j^*(\mathbf{u}_i^k)^T \Delta_{k+1}\mathbf{v}_j^k + \sigma_i^*(\mathbf{u}_j^k)^T \Delta_{k+1}\mathbf{v}_i^k + \sigma_j^*[R_k]_{ij} + \sigma_i^*[R_k]_{ji}}{(\sigma_j^*)^2 - (\sigma_i^*)^2},\ s+1 \le i \le n-t,\ 1 \le j \le n-t, i > j. \end{aligned} \tag{3.18}$$

where and in sequel $\Delta_{k+1} := A(\mathbf{c}^*) - A(\mathbf{c}^{k+1})$. Note that $\{\mathbf{u}_i^k\}_{i=1}^n$ and $\{\mathbf{v}_i^k\}_{i=1}^n$ are orthonormal and that, by Lemma 2.3,

$$\|\Delta_{k+1}\| \le \alpha \|\mathbf{c}^{k+1} - \mathbf{c}^*\|.$$

One has by (3.17) and (3.18) that

$$|[\tilde{X}_k]_{ij} - [X_k]_{ij}| \le \frac{1}{\sigma_j^*}\left(\alpha\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|R_k\|_F\right), \quad n-t+1 \le i \le n,\ 1 \le j \le n-t, \tag{3.19}$$

$$|[\tilde{X}_k]_{ij} - [X_k]_{ij}| \le \frac{1}{\sigma_j^* - \sigma_i^*}\left(\alpha\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|R_k\|_F\right), \quad s+1 \le i \le n-t,\ 1 \le j \le n-t,\ i > j. \tag{3.20}$$

Since $[X_k]_{ij} = 0$ for each $1 \le i,\ j \le s$ or $n-t+1 \le i,\ j \le n$, we have by (2.1), (3.19), (3.20) and the definition of $\eta_1$ that

$$\|\tilde{X}_k - X_k\| \le \|\tilde{X}_k - X_k\|_F \le \|\tilde{X}_k^{[s]_U}\|_F + \|\tilde{X}_k^{[t]_L}\|_F + \eta_1\left(\alpha\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|R_k\|_F\right),$$

Combining this with (3.6) and (3.14), we further derive that

$$\|\tilde{X}_k - X_k\| \le 2\gamma_1\|E_{U_k}\|^2 + \eta_1\left[\alpha\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + 4\gamma_1^2\|\boldsymbol{\sigma}^*\|(\|E_{U_k}\|^2 + \|E_{V_k}\|^2)\right], \tag{3.21}$$

$$\|X_k\| \le \gamma_1\|E_{U_k}\| + 2\gamma_1\|E_{U_k}\|^2 + \eta_1\left[\alpha\|\mathbf{c}^{k+1}-\mathbf{c}^*\| + 4\gamma_1^2\|\boldsymbol{\sigma}^*\|(\|E_{U_k}\|^2+\|E_{V_k}\|^2)\right].$$

Thus, by the fact of $\gamma_1\max\{\|E_{U_k}\|,\ \|E_{V_k}\|\}\le\gamma_1\delta<1$, one has

$$\|X_k\| \le \eta_1\alpha\|\mathbf{c}^{k+1}-\mathbf{c}^*\| + (2+\gamma_1+4\gamma_1\eta_1\|\boldsymbol{\sigma}^*\|)\|E_{U_k}\| + 4\gamma_1\eta_1\|\boldsymbol{\sigma}^*\|\cdot\|E_{V_k}\|; \tag{3.22}$$

hence,

$$\|X_k\| \le \frac{\eta_2}{2}\left(\|\mathbf{c}^{k+1}-\mathbf{c}^*\| + \|E_{U_k}\| + \|E_{V_k}\|\right) \le \frac{\gamma}{2}\left(\|\mathbf{c}^{k+1}-\mathbf{c}^*\| + \|E_{U_k}\| + \|E_{V_k}\|\right) \tag{3.23}$$

(noting that $\gamma \ge \eta_2 = \max\{2\eta_1\alpha,\ 4+2\gamma_1+8\gamma_1\eta_1\|\boldsymbol{\sigma}^*\|\}$). Since $\max\{\|E_{U_k}\|,\ \|E_{V_k}\|\}\le\delta$ and $\|\mathbf{c}^{k+1}-\mathbf{c}^*\|\le\delta$, we derive further by (3.9) and (3.23) that $\|X_k\|\le 1$. Therefore, applying Lemma 2.1 (for $A=I-\frac{1}{2}X_k$ and $B=I$), one has

$$\left\|\left(I-\tfrac{1}{2}X_k\right)^{-1}\right\| \le \frac{1}{1-\frac{1}{2}\|X_k\|} \le 2. \tag{3.24}$$

Consequently, the estimates of $\|\tilde{X}_k - X_k\|$, $\|X_k\|$, and $\left\|\left(I-\frac{1}{2}X_k\right)^{-1}\right\|$ are complete. By a similar argument, we can have the following estimates:

$$\|\tilde{Y}_k - Y_k\| \le 2\gamma_1\|E_{V_k}\|^2 + \eta_1\left[\alpha\|\mathbf{c}^{k+1}-\mathbf{c}^*\| + 4\gamma_1^2\|\boldsymbol{\sigma}^*\|(\|E_{U_k}\|^2+\|E_{V_k}\|^2)\right],$$

$$\|Y_k\| \le \frac{\gamma}{2}\left(\|\mathbf{c}^{k+1}-\mathbf{c}^*\| + \|E_{U_k}\| + \|E_{V_k}\|\right), \tag{3.25}$$

and

$$\left\|\left(I-\tfrac{1}{2}Y_k\right)^{-1}\right\| \le 2. \tag{3.26}$$

Now we offer the estimates of $\|U_{k+1}-U_k\|$, $\|V_{k+1}-V_k\|$, $\|E_{U_{k+1}}\|$, and $\|E_{V_{k+1}}\|$. Note by (3.2) that

$$U_{k+1}-U_k = U_k[(I+\tfrac{1}{2}X_k)-(I-\tfrac{1}{2}X_k)](I-\tfrac{1}{2}X_k)^{-1} = U_kX_k(I-\tfrac{1}{2}X_k)^{-1}.$$

This together with (3.23), (3.24), and the orthonormal property of $U_k$ gives rise to

$$\|U_{k+1}-U_k\| \le \gamma\left(\|\mathbf{c}^{k+1}-\mathbf{c}^*\| + \|E_{U_k}\| + \|E_{V_k}\|\right).$$

Similarly, using (3.3), (3.25), and (3.26), we obtain

$$\|V_{k+1}-V_k\| \le \gamma\left(\|\mathbf{c}^{k+1}-\mathbf{c}^*\| + \|E_{U_k}\| + \|E_{V_k}\|\right).$$

Thus, assertion (ii) holds. It remains to prove assertion (iii). The arguments for the estimates of $E_{U_{k+1}}$ and $E_{V_{k+1}}$ are similar and so we only estimate $E_{U_{k+1}}$. For this, note by (3.2) and the definition of $\tilde{X}_k$ that

$$U_{k+1}-\tilde{U}(\mathbf{c}^*) = U_k[(I+\tfrac{1}{2}X_k)(I-\tfrac{1}{2}X_k)^{-1} - e^{\tilde{X}_k}] = U_k[(I+\tfrac{1}{2}X_k) - e^{\tilde{X}_k}(I-\tfrac{1}{2}X_k)](I-\tfrac{1}{2}X_k)^{-1}.$$

Then, using the equality $e^{\tilde{X}_k} = \sum\limits_{m=0}^{\infty}\frac{\tilde{X}_k^m}{m!}$, it is easy to check that

$$\begin{aligned} U_{k+1}-\tilde{U}(\mathbf{c}^*) &= U_k\Big[X_k-\tilde{X}_k+\tfrac{1}{2}\tilde{X}_kX_k-\Big(\tilde{X}_k^2\sum_{m=2}^{\infty}\frac{\tilde{X}_k^{m-2}}{m!}\Big)\Big(I-\frac{1}{2}X_k\Big)\Big]\Big(I-\frac{1}{2}X_k\Big)^{-1} \\ &= U_k(X_k-\tilde{X}_k)(I-\tfrac{1}{2}X_k)^{-1} + \tfrac{1}{2}U_k\tilde{X}_kX_k(I-\tfrac{1}{2}X_k)^{-1} - U_k\tilde{X}_k^2\sum_{m=2}^{\infty}\frac{\tilde{X}_k^{m-2}}{m!}. \end{aligned} \tag{3.27}$$

Noting that $U_k$ is orthonormal, we deduce from (3.27), (3.10), (3.24), and (2.1) that

$$\|U_{k+1}-\tilde{U}(\mathbf{c}^*)\| \le 2\|X_k-\tilde{X}_k\|+\|\tilde{X}_k\|\cdot\|X_k\|+\|\tilde{X}_k\|^2 \le 2\|X_k-\tilde{X}_k\|+\|\tilde{X}_k\|_F\cdot\|X_k\|+\|\tilde{X}_k\|_F^2.$$

Thus, one has by (3.6), (3.21), and (3.23) that

$$\begin{array}{rcl}\|U_{k+1}-\tilde{U}(\mathbf{c}^*)\| & \le & (4\gamma_1+\gamma_1^2+8\gamma_1^2\eta_1\|\boldsymbol{\sigma}^*\|+\frac{1}{2}\gamma_1\eta_2)(\|E_{U_k}\|+\|E_{V_k}\|)^2 \\ & & +(2\eta_1\alpha+\frac{1}{2}\eta_2)\|\mathbf{c}^{k+1}-\mathbf{c}^*\|.\end{array} \tag{3.28}$$

Recall from (3.8) that $\eta_2=\max\{2\eta_1\alpha,\ 4+2\gamma_1+8\gamma_1\eta_1\|\boldsymbol{\sigma}^*\|\}$. We then have by (3.28) that

$$\|U_{k+1}-\tilde{U}(\mathbf{c}^*)\| \le \frac{3}{2}\gamma_1\eta_2(\|E_{U_k}\|+\|E_{V_k}\|)^2+\frac{3}{2}\eta_2\|\mathbf{c}^{k+1}-\mathbf{c}^*\| \le \frac{1}{3}\gamma[\|\mathbf{c}^{k+1}-\mathbf{c}^*\|+(\|E_{U_k}\|+\|E_{V_k}\|)^2]. \tag{3.29}$$

where the last inequality holds because, by the definitions of $\gamma$ and $\gamma_1$, $\gamma \ge \frac{9}{2}\gamma_1\eta_2 \ge \frac{9}{2}\eta_2$. To proceed, write $U_{k+1}:=[U_{k+1}^{(1)},\ U_{k+1}^{(2)},\ U_{k+1}^{(3)}]$ where $U_{k+1}^{(1)}\in\mathbb{R}^{n\times s}$ and $U_{k+1}^{(3)}\in\mathbb{R}^{n\times t}$. Since $(I-\Pi_{U,i})\tilde{U}^{(i)}(\mathbf{c}^*)=\mathbf{0}$ and $\|I-\Pi_{U,i}\|\le 1$ hold for $i=1,\ 3$, one has

$$\|(I-\Pi_{U,i})U_{k+1}^{(i)}\| = \|(I-\Pi_{U,i})(U_{k+1}^{(i)}-\tilde{U}^{(i)}(\mathbf{c}^*))\| \le \|U_{k+1}-\tilde{U}(\mathbf{c}^*)\|, \quad i=1,\ 3. \tag{3.30}$$

Noting that $E_{U_{k+1}}=[(I-\Pi_{U,1})U_{k+1}^{(1)},\ U_{k+1}^{(2)}-U^{(2)}(\mathbf{c}^*),\ (I-\Pi_{U,3})U_{k+1}^{(3)}]$, we obtain from (2.1), (3.29) and (3.30) that

$$\begin{array}{rcl}\|E_{U_{k+1}}\| & \le & \|(I-\Pi_{U,1})U_{k+1}^{(1)}\|_F+\|U_{k+1}^{(2)}-U^{(2)}(\mathbf{c}^*)\|_F+\|(I-\Pi_{U,3})U_{k+1}^{(3)}\|_F \\ & \le & 3\sqrt{n}\|U_{k+1}-\tilde{U}(\mathbf{c}^*)\|.\end{array} \tag{3.31}$$

Therefore, thanks to (3.29) and (3.31), one sees that

$$\|E_{U_{k+1}}\| \le \gamma[\|\mathbf{c}^{k+1}-\mathbf{c}^*\|+(\|E_{U_k}\|+\|E_{V_k}\|)^2];$$

hence, assertion (iii) holds. The proof is complete. □

Now we present the main result of this paper which shows that the Ulm-like method proposed here converges quadratically for solving the square ISVP. Recall from Lemma 2.8 that $\partial_{Q|\mathcal{S}}\mathbf{f}(\mathbf{c}^*)$ is nonempty if $\mathbf{c}^*\in\mathrm{cl}\mathcal{S}$.

**Theorem 3.1.** *Let $\mathbf{c}^*\in\mathrm{cl}\mathcal{S}$ such that the matrix $A(\mathbf{c}^*)$ has singular values given by* (1.6)*. Suppose that each $J\in\partial_{Q|\mathcal{S}}\mathbf{f}(\mathbf{c}^*)$ is nonsingular. Then there exist $\delta\in(0,\ 1)$ and $\mu\in(0,\ 1)$ such that for each $\mathbf{c}^0\in\mathbf{B}(\mathbf{c}^*,\delta)\cap\mathcal{S}$ and each $B_0$ satisfying*

$$\|I-B_0J_0\|\le\mu, \tag{3.32}$$

*the sequence $\{\mathbf{c}^k\}$ generated by the Ulm-like method with initial point $\mathbf{c}^0$ converges quadratically to $\mathbf{c}^*$.*

*Proof.* By Lemma 3.1, let $\delta_1\in(0,\ 1)$ and $\gamma\in[1,\ +\infty)$ such that for any $k\ge 0$ and $[U(\mathbf{c}^*)\ V(\mathbf{c}^*)]\in\mathcal{W}(\mathbf{c}^*)$, if $\max\{\|E_{U_k}\|,\ \|E_{V_k}\}<\delta_1$, the assertions (i)–(iii) in Lemma 3.1 hold with $\delta=\delta_1$. Moreover, thanks to Lemmas 2.6 and 2.10, we assume without loss of generality that for any $\mathbf{c}\in\mathbf{B}(\mathbf{c}^*,\delta_1)$, (2.35) and the assertions (i)–(iv) in Lemma 2.6 hold. Write

$$q:=6\sqrt{n}\gamma.$$

Take $\delta$ and $\mu$ such that

$$0<\delta<\min\left\{\frac{2\delta_1}{q},\ \frac{1}{q\gamma(5+8\sqrt{n}\gamma^2)},\ \frac{1}{q(2+1152n^2\gamma^4\cdot\max\limits_j\|A_j\|)}\right\},\quad 0\le\mu\le\delta. \tag{3.33}$$

Clearly, $\delta\in(0,\ 1)$ and $\mu\in(0,\ 1)$. Below we shall show that $\delta$ and $\mu$ are as desired. For this purpose, let $\mathbf{c}^0\in\mathbf{B}(\mathbf{c}^*,\delta)\bigcap S$ and $B_0$ satisfy $\|I-B_0J_0\|\le\mu$. Then, thanks to Lemma 2.9 and the definition of $J_0$, one has that $\partial_{Q|S}\mathbf{f}(\mathbf{c}^0)=\{\mathbf{f}'(\mathbf{c}^0)\}=\{J_0\}$. In addition, by Lemma 2.10 (as $\delta<\frac{2\delta_1}{q}\le\delta_1$), we have

$$\|J_0^{-1}\|\le\gamma. \tag{3.34}$$

It suffices to prove that for any $k=0,\ 1,\ \ldots,$

$$\|\mathbf{c}^k-\mathbf{c}^*\|\le q\delta\left(\frac{1}{2}\right)^{2^k}, \tag{3.35}$$

$$\max\{\|E_{U_k}\|,\ \|E_{V_k}\|\}\le q\delta\left(\frac{1}{2}\right)^{2^k}, \tag{3.36}$$

and

$$\|I-B_kJ_k\|\le q\delta\left(\frac{1}{2}\right)^{2^k}. \tag{3.37}$$

We proceed by mathematical induction. Since $\|\mathbf{c}^0-\mathbf{c}^*\|<\delta$, $\|I-B_0J_0\|\le\mu\le\delta$ and $q\ge 2$, (3.35) and (3.37) are trivial for $k=0$. Noting that $E_{U_0}=[(I-\Pi_{U,1})U_0^{(1)},\ U_0^{(2)}-U^{(2)}(\mathbf{c}^*),\ (I-\Pi_{U,3})U_0^{(3)}]$, one has by (2.1) and Lemma 2.6 that

$$\|E_{U_0}\|\le\|E_{U_0}\|_F\le\|(I-\Pi_{U,1})U_0^{(1)}\|_F+\|U_0^{(2)}-U^{(2)}(\mathbf{c}^*)\|_F+\|(I-\Pi_{U,3})U_0^{(3)}\|_F\le 3\sqrt{n}\gamma\delta=\frac{1}{2}q\delta,$$

where the equality holds because of the definition of $q$. Similarly, one can prove that $\|E_{V_0}\|\le\frac{1}{2}q\delta$; hence, (3.36) is shown for $k=0$. Assume that (3.35)–(3.37) hold for all $k\le m$. Then, by (3.33),

$$\|\mathbf{c}^k-\mathbf{c}^*\|\le\frac{1}{2}q\delta<\delta_1\quad\text{and}\quad\max\{\|E_{U_k}\|,\ \|E_{V_k}\|\le\frac{1}{2}q\delta<\delta_1,\quad\text{for each }k\le m. \tag{3.38}$$

Thus, applying Lemma 3.1, we get that

$$\|\Sigma^*+\tilde{X}_m\Sigma^*-\Sigma^*\tilde{Y}_m-U_m^TA(\mathbf{c}^*)V_m\|\le\gamma(\|E_{U_m}\|^2+\|E_{V_m}\|^2) \tag{3.39}$$

and

$$\max\{\|U_{k+1}-U_k\|,\ \|V_{k+1}-V_k\|\}\le\gamma(\|\mathbf{c}^{k+1}-\mathbf{c}^*\|+\|E_{U_k}\|+\|E_{V_k}\|),\quad\text{for each }k\le m-1. \tag{3.40}$$

Considering the diagonal entries of $\Sigma^*+\tilde{X}_m\Sigma^*-\Sigma^*\tilde{Y}_m-U_m^TA(\mathbf{c}^*)V_m$, one sees from (3.39) that

$$|(\mathbf{u}_i^m)^TA(\mathbf{c}^*)\mathbf{v}_i^m-\sigma_i^*|\le\gamma(\|E_{U_m}\|^2+\|E_{V_m}\|^2),\quad\text{for each }1\le i\le n.$$

Therefore, by the definitions of $\boldsymbol{\sigma}^*$, $J_m$, $\mathbf{b}^m$, and $A(\mathbf{c}^*)$,

$$\|J_m\mathbf{c}^*+\mathbf{b}^m-\boldsymbol{\sigma}^*\|\le\sqrt{n}\gamma(\|E_{U_m}\|^2+\|E_{V_m}\|^2). \tag{3.41}$$

Noting that

$$\|U_m - U_0\| \le \sum_{k=0}^{m-1} \|U_{k+1} - U_k\| \quad \text{and} \quad \|V_m - V_0\| \le \sum_{k=0}^{m-1} \|V_{k+1} - V_k\|,$$

we have by (3.40), (3.35) (with $k \le m-1$), and (3.36) (with $k \le m-1$) that

$$\max\{\|U_m - U_0\|,\ \|V_m - V_0\|\} \le 3\gamma q\delta \left[\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^{2^2} + \cdots + \left(\frac{1}{2}\right)^{2^{m-1}}\right].$$

Since $2^n \ge n+1$ for each $n \ge 0$, it follows that

$$\max\{\|U_m - U_0\|,\ \|V_m - V_0\|\} \le 3\gamma q\delta \left[\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^m\right] \le 3\gamma q\delta.$$

This together with (3.33) yields

$$2n\gamma \cdot \max_j \|A_j\| \cdot \max\{\|U_m - U_0\|,\ \|V_m - V_0\|\} \le 6n\gamma^2 q\delta \cdot \max_j \|A_j\| < \frac{1}{2}, \tag{3.42}$$

where the last inequality holds because, by the definition of $\delta$ and the fact of $\gamma \ge 1$,

$$12n\gamma^2 q\delta \cdot \max_j \|A_j\| < q\delta(2 + 1152n^2\gamma^4 \cdot \max_j \|A_j\|) \le 1.$$

Moreover, by the definition of $[J_m]_{ij}$ (cf. (3.5)), one has

$$\left|[J_m]_{ij} - [J_0]_{ij}\right| = \left|(\mathbf{u}_i^m - \mathbf{u}_i^0)^T A_j \mathbf{v}_i^m + (\mathbf{u}_i^0)^T A_j (\mathbf{v}_i^m - \mathbf{v}_i^0)\right| \le 2\|A_j\| \cdot \max\{\|\mathbf{u}_i^m - \mathbf{u}_i^0\|,\ \|\mathbf{v}_i^m - \mathbf{v}_i^0\|\}.$$

Then, thanks to (2.1), we get that

$$\|J_m - J_0\| \le \|J_m - J_0\|_F \le 2n \max_j \|A_j\| \cdot \max\{\|U_m - U_0\|,\ \|V_m - V_0\|\}.$$

It follows from (3.34) and (3.42) that

$$\|J_0^{-1}\| \cdot \|J_m - J_0\| \le 2n\gamma \cdot \max_j \|A_j\| \cdot \max\{\|U_m - U_0\|,\ \|V_m - V_0\|\} < \frac{1}{2}.$$

Thus, applying Lemma 2.1 (for $A = J_m$ and $B = J_0$) and using (3.34) again, we obtain

$$\|J_m^{-1}\| \le \frac{\|J_0^{-1}\|}{1 - \|J_0^{-1}\| \cdot \|J_m - J_0\|} < 2\gamma. \tag{3.43}$$

On the other hand, by using the inductive assumption of (3.37) (with $k = m$), one has

$$\|B_m J_m\| = \|I - B_m J_m - I\| \le 1 + \|I - B_m J_m\| \le 1 + q\delta \left(\frac{1}{2}\right)^{2^m}.$$

This and (3.43) give rise to

$$\|B_m\| \le \|B_m J_m\| \cdot \|J_m^{-1}\| < 2\gamma \left[1 + q\delta \left(\frac{1}{2}\right)^{2^m}\right] \le 4\gamma \tag{3.44}$$

(noting that $q\delta \le 1$ by (3.33)). Now, we show that (3.35)–(3.37) hold for $k = m+1$. By (3.1) (with $k = m$),

$$\begin{aligned} \mathbf{c}^{m+1} - \mathbf{c}^* &= \mathbf{c}^m - \mathbf{c}^* - B_m J_m \mathbf{c}^m - B_m \mathbf{b}^m + B_m \boldsymbol{\sigma}^* \\ &= (I - B_m J_m)(\mathbf{c}^m - \mathbf{c}^*) - B_m(J_m \mathbf{c}^* + \mathbf{b}^m - \boldsymbol{\sigma}^*). \end{aligned}$$

Then, by (3.41), (3.44), and the inductive assumptions with $k = m$, we obtain

$$\begin{aligned} \|\mathbf{c}^{m+1} - \mathbf{c}^*\| &\le \|I - B_m J_m\| \cdot \|\mathbf{c}^m - \mathbf{c}^*\| + \|B_m\| \cdot \|J_m \mathbf{c}^* - \boldsymbol{\sigma}^* + \mathbf{b}^m\| \\ &\le (q\delta)^2 \left(\frac{1}{2}\right)^{2^{m+1}} + 8\sqrt{n}\gamma^2 (q\delta)^2 \left(\frac{1}{2}\right)^{2^{m+1}}. \end{aligned} \tag{3.45}$$

Thus, (3.35) holds for $k = m+1$ and moreover $\|\mathbf{c}^{m+1} - \mathbf{c}^*\| \le \delta_1$ as $q\delta \le \min\{2\delta_1,\ 1/(1 + 8\sqrt{n}\gamma^2)\}$ by (3.33). Hence, noting (3.38), Lemma 3.1 (ii) and (iii) (with $k = m$) are applicable to concluding that

$$\max\{\|U_{m+1} - U_m\|,\ \|V_{m+1} - V_m\|\} \le \gamma(\|\mathbf{c}^{m+1} - \mathbf{c}^*\| + \|E_{U_m}\| + \|E_{V_m}\|) \tag{3.46}$$

and

$$\max\{\|E_{U_{m+1}}\|,\ \|E_{V_{m+1}}\|\} \le \gamma[\|\mathbf{c}^{m+1} - \mathbf{c}^*\| + (\|E_{U_m}\| + \|E_{V_m}\|)^2]$$

Therefore we derive from (3.45) and the inductive assumption (3.36) (with $k = m$) that

$$\max\{\|E_{U_{m+1}}\|,\ \|E_{V_{m+1}}\|\} \le \gamma \left[5(q\delta)^2 \left(\frac{1}{2}\right)^{2^{m+1}} + 8\sqrt{n}\gamma^2 (q\delta)^2 \left(\frac{1}{2}\right)^{2^{m+1}}\right].$$

Thus, (3.36) holds for $k = m+1$ because of $\gamma(5 + 8\sqrt{n}\gamma^2) q\delta \le 1$ by (3.33). On the other hand, thanks to (3.46), (3.35) (with $k = m+1$), and (3.36) (with $k = m+1$) just proved, one can see that

$$\max\{\|U_{m+1} - U_m\|,\ \|V_{m+1} - V_m\|\} \le 3\gamma q\delta \left(\frac{1}{2}\right)^{2^m},$$

which implies

$$\max\{\|\mathbf{u}_i^{m+1} - \mathbf{u}_i^m\|,\ \|\mathbf{v}_i^{m+1} - \mathbf{v}_i^m\|\} \le 3\gamma q\delta \left(\frac{1}{2}\right)^{2^m}, \quad \text{for each} \quad 1 \le i \le n.$$

Consequently, for any $1 \le i,\ j \le n$,

$$\begin{aligned} |\ [J_{m+1}]_{ij} - [J_m]_{ij}\ | &= |\ (\mathbf{u}_i^{m+1} - \mathbf{u}_i^m)^T A_j \mathbf{v}_i^{m+1} - (\mathbf{u}_i^m)^T A_j (\mathbf{v}_i^m - \mathbf{v}_i^{m+1})\ | \\ &\le 2 \max_j \|\ A_j\ \| \cdot \max\{\|\mathbf{u}_i^{m+1} - \mathbf{u}_i^m\|,\ \|\mathbf{v}_i^{m+1} - \mathbf{v}_i^m\|\} \\ &\le 6\gamma q\delta \cdot \max_j \|\ A_j\ \| \left(\frac{1}{2}\right)^{2^m}; \end{aligned}$$

hence

$$\|J_{m+1} - J_m\| \le \|J_{m+1} - J_m\|_F \le 6n\gamma q\delta \cdot \max_j \|\ A_j\ \| \left(\frac{1}{2}\right)^{2^m}. \tag{3.47}$$

Since $B_k = 2B_{k-1} - B_{k-1} J_k B_{k-1}$ for each $k = 1, 2, \ldots$, one has

$$I - B_{m+1} J_{m+1} = (I - B_m J_{m+1})^2 = [I - B_m J_m - B_m (J_{m+1} - J_m)]^2.$$

Then, by (3.44), (3.47) and using the inductive assumption (3.37) with $k = m$, we derive

$$\begin{aligned}\|I - B_{m+1}J_{m+1}\| &\leq 2\|I - B_mJ_m\|^2 + 2\|B_m\|^2 \cdot \|J_{m+1} - J_m\|^2 \\ &\leq \left(2 + 1152n^2\gamma^4 \cdot \max_j \|A_j\|^2\right)(q\delta)^2\left(\frac{1}{2}\right)^{2^{m+1}}.\end{aligned} \tag{3.48}$$

Note by (3.33) that $(2 + 1152n^2\gamma^4 \cdot \max_j \|A_j\|^2)q\delta \leq 1$. It follows from (3.48) that

$$\|I - B_{m+1}J_{m+1}\| \leq q\delta\left(\frac{1}{2}\right)^{2^{m+1}}.$$

This verifies (3.37) holding for $k = m + 1$ and the proof is complete.

□

Theorems 3.2 and 3.3 below, the proofs of which are similar to that of Theorems 3.1, show that the condition $\mathbf{c}^* \in \mathrm{cl}\mathcal{S}$ is not required if the nonsingularity assumption for each $J \in \partial_{Q|\mathcal{S}}\mathbf{f}(\mathbf{c}^*)$ is replaced by the nonsingularity assumption for each $J \in \partial_Q\mathbf{f}(\mathbf{c}^*)$ or each $J \in \partial_B\mathbf{f}(\mathbf{c}^*)$.

**Theorem 3.2.** *Let $\mathbf{c}^* \in \mathbb{R}^n$ be such that the matrix $A(\mathbf{c}^*)$ has singular values given by* (1.6). *Suppose that each $J \in \partial_Q\mathbf{f}(\mathbf{c}^*)$ is nonsingular. Then there exist $\delta \in (0,\ 1)$ and $\mu \in (0,\ 1)$ such that for each $\mathbf{c}^0 \in \mathbf{B}(\mathbf{c}^*, \delta)\bigcap\mathcal{S}$ and each $B_0$ satisfying* (3.32)*, the sequence $\{\mathbf{c}^k\}$ generated by the Ulm-like method with initial point $\mathbf{c}^0$ converges quadratically to $\mathbf{c}^*$.*

**Theorem 3.3.** *Let $\mathbf{c}^* \in \mathbb{R}^n$ be such that the matrix $A(\mathbf{c}^*)$ has singular values given by* (1.6). *Suppose that each $J \in \partial_B\mathbf{f}(\mathbf{c}^*)$ is nonsingular. Then there exist $\delta \in (0,\ 1)$ and $\mu \in (0,\ 1)$ such that for each $\mathbf{c}^0 \in \mathbf{B}(\mathbf{c}^*, \delta) \cap \mathcal{S}$ and each $B_0$ satisfying* (3.32)*, the sequence $\{\mathbf{c}^k\}$ generated by the Ulm-like method with initial point $\mathbf{c}^0$ converges quadratically to $\mathbf{c}^*$.*

# 4 Numerical tests

In this section, we report some numerical tests to illustrate the convergence performance of the Ulm-like method. In all tests, multiple and zero singular values are present in the given singular values. Our aim is, for the inverse singular value problems with multiple and zero singular values, to illustrate the validity of the Ulm-like method. All the tests were implemented in MATLAB 7.0 on a Genuine Intel(R) PC with 1.6 GHz CPU.

Let $\{T_i\}_{i=1}^n$ be Toeplitz matrices given by

$$T_1 = I, \quad T_2 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix}, \quad \ldots, \quad T_n = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

Let $H_1 \subset \mathcal{O}(n)$, $H_2 \subset \mathcal{O}(n)$ be orthonormal bases for the range of $R1 \subset \mathbb{R}^{n\times n}$ and $R2 \subset \mathbb{R}^{n\times n}$ respectively which are generated by Matlab-provided randn function. Define $A_0 = \mathbf{0}$ and $\{A_i\}_{i=1}^n \subset \mathbb{R}^{m\times n}$ as follows

$$A_i = H_1T_iH_2, \quad \text{for each } i = 1, 2, \ldots, n,$$

Here we focus on the following cases: $50 \times 50$, $100 \times 100$, and $200 \times 200$. To present multiple and zero singular values, we first generate in each test a vector $\hat{\mathbf{c}}^*$ randomly such that there exist integers $p$ and $q$ such that the singular values of matrix $A(\tilde{\mathbf{c}}^*)$ satisfying $|\sigma_{p+1}(\tilde{\mathbf{c}}^*) - \sigma_p(\tilde{\mathbf{c}}^*)| < 5e-5$ and $\sigma_q(\tilde{\mathbf{c}}^*) < 1e-4$, where $\tilde{\mathbf{c}}^* := \hat{\mathbf{c}}^* * 10^{-4}$. Set

$$\sigma_i^* = \begin{cases} \sigma_p(\tilde{\mathbf{c}}^*), & i = p,\ p+1; \\ 0, & i = q; \\ \sigma_i(\tilde{\mathbf{c}}^*), & \text{otherwise}. \end{cases}$$

Then we choose $\{\sigma_i^*\}_{i=1}^n$ as the prescribed singular values.

Since the Newton-type method is locally convergent, $\mathbf{c}^0$ is formed by chopping the components of $\tilde{\mathbf{c}}^*$ to five decimal places for the cases of $50 \times 50$, $100 \times 100$, and to six decimal places for the cases of $200 \times 200$. Note by (3.32) that $B_0$ is an approximation to $J_0$. For simplicity, here we take $B_0$ by perturbing $J_0^{-1}$. However, in the case when $J_0^{-1}$ is difficult to obtained, one may set $B_0 = [\mathbf{z}_1,\ \ldots,\ \mathbf{z}_n]^T$ where for each $1 \le i \le n$, $\mathbf{z}_i$ is an approximate solution of the equation

$$J_0^T \mathbf{x} = \mathbf{e}_i$$

such that $\|\mathbf{e}_i - J_0^T \mathbf{z}_i\| \le \mu/n$ (note that with such choice of $B_0$, (3.32) is seen to hold). Here $\mathbf{e}_i$ is the $i$-th column of the identity matrix $I$. In all test problems, systems (3.2) and (3.3) were solved by the QMR method [9] via the MATLAB QMR function, where the maximal number of iterations is set to be 1000. Also, for these two systems, we use the right-hand side vector as the initial guess. Moreover, to guarantee the orthogonality of $U_k$ and $V_k$, systems (3.2) and (3.3) are solved up to machine precision eps. Finally, the outer iteration for the Ulm-like method is stopped when

$$\|U_k^T A(\mathbf{c}^k) V_k - \Sigma^*\|_F < 10^{-13}.$$

We now report our experimental results. Table 1 illustrates the values of $d_k := \|U_k^T A(\mathbf{c}^k) V_k - \Sigma^*\|_F$ for all test problems. For simplicity, we only choose $\mu = 0.001$ in Table 1. To further illustrate the influence of $\mu$ to the convergence performance of the Ulm-like method, we present the numerical results for different $\mu$s in Table 2 where the values of $d_k$ for the problem of size $50 \times 50$ are illustrated. We can see from Tables 1 and 2 that the Ulm-like method converges superlinearly. This confirms the theoretical results of our paper. Moreover, the convergence performances the Ulm-like method with $\mu \le 0.01$ are comparable to that of the Ulm-like method with $\mu = 0$. Thus, the Ulm-like method with small $\mu$ is more effective than that with large $\mu$.

# 5 Conclusions

Noting the interesting problem rased in [SIAM J. Matrix Anal. Appl. 32 (2011), 412–429], we proposed in this paper an Ulm-like method for solving the square inverse singular value problems (a special case of the inverse singular value problems ) with multiple and zero singular values, and proved that it converges at least quadratically under some nonsingularity assumptions. However, we don't know whether the proposed method is still work for the inverse singular value problems which needs further study.

Table 1: The values of $d_k$ with $\mu = 0.001$

| $k$ | $50 \times 50$ | $100 \times 100$ | $200 \times 200$ |
|---|---|---|---|
| 0 | $3.08e-04$ | $6.04e-05$ | $2.26e-04$ |
| 1 | $5.39e-06$ | $9.39e-06$ | $1.21e-05$ |
| 2 | $1.21e-07$ | $1.30e-07$ | $3.23e-07$ |
| 3 | $9.02e-11$ | $6.91e-11$ | $1.29e-10$ |
| 4 | $1.03e-16$ | $1.12e-16$ | $7.90e-17$ |
| 5 | $\vdots$ | $\vdots$ | $\vdots$ |

Table 2: Values of $d_k$ for different $\mu$s.

| $d_k$ | $\mu = 0$ | $\mu = 0.0001$ | $\mu = 0.001$ | $\mu = 0.01$ | $\mu = 0.05$ | $\mu = 0.1$ |
|---|---|---|---|---|---|---|
| $d_0$ | $3.42e-04$ | $3.42e-04$ | $3.42e-04$ | $3.42e-04$ | $3.42e-04$ | $3.42e-04$ |
| $d_1$ | $8.10e-06$ | $8.04e-06$ | $7.92e-06$ | $2.66e-05$ | $1.35e-04$ | $2.71e-04$ |
| $d_2$ | $6.42e-08$ | $6.39e-08$ | $6.17e-08$ | $9.64e-08$ | $1.02e-05$ | $8.15e-05$ |
| $d_3$ | $4.76e-12$ | $4.72e-12$ | $4.30e-12$ | $3.84e-12$ | $1.25e-7$ | $1.69e-05$ |
| $d_4$ | $3.00e-18$ | $1.00e-18$ | $1.00e-18$ | $2.00e-18$ | $4.77e-08$ | $2.01e-06$ |
| $d_5$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $1.20e-17$ | $4.42e-08$ |
| $d_6$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $2.12e-11$ |
| $d_7$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $5.00e-18$ |
| $d_8$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

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